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Generalized solutions of Nonlinear Systems with the ”Generalized Iterative Method approximation of Root (GRIM)”

Nikos Mantzakouras & Eteri Biragova
e-mail: nikmatza@gmail.com
Athens – Greece

Abstract

Solving nonlinear equations in any Banach space (real or complex nonlinear equations, nonlinear systems, and nonlinear matrix equations, among others) is a non-trivial task involving many areas of science and technology. Usually the solution is not readily accessible and requires an approach that uses iterative algorithms. In the present work, which is complementary to previously published work, the focus is mainly on the design, convergence analysis and stability of new iterative schemes for solving nonlinear problems and their application to scientifically important problems. The methods we will use are two. One involves the well-known Newton with Jacobi-matrix method, but with the help of the generalized solution theorem [1] & [3]. The second, that is new, deals with solving equations using the GRIM method, which is an iterative method and which is again based on the generalized theorem solve equations [1] & [3]. The potential of the generalized theorem is that it gives us the maximum number of real or complex roots by a categorization procedure based on integer coefficients in general, rather than a stepwise or random search in some the interval.

Keywords: *Nonlinear Systems, polynomial and transcendental equations, Newton Jacobi method, iterative Fixed Point Method, initial values of roots.*

I. Newton Method with Jacobian-matrix to resolve nonlinear systems.

The NEWTON method, so-called Newton-Raphson method, is one a widely used repeated method for finding the roots of a non-linear equations systems. This method is due to the English mathematician, fysicist and astronomer Sir Isaac Newton (1642-1727). The method was published in 1685 (John Wallis, A Treatise and Practical), while in 1690 there was a simplified description of the method by Joseph Raphson (Joseph Raphson, Analysis Aequationum Universalis). The repetitive shape of the method is described later in 1740 by Thomas Simpson. In the same publication, Thomas Simpson generalizes the method for solving a system of two equations, as it also points out that the method also applies to optimization problems. Newton method repeatedly approaches the root of a non-linear actual function. It is also generalized to solve a system of non-linear algebraic equations. The method in combination with the Jacobian matrix, it is useful to resolve of non-linear systems equations.

I.1. This problem is defined as follows

Is given $F_n = (f_1, f_2, \dots, f_n) : D_n \subset C^n \to C^n$ to be found $x^r \in D_n : F_n(x_r) = \theta_n$ where $\theta_n = (0, 0, .., 0)^T$ and where $f_i(x)$ for $x = (x_1, x_2, .., x_n)^T \in C^n$ denotes the i-position component of F_n. The equation $F_n(x_r) = \theta_n$ written

$$F_n\left(x_r\right)=\left\langle\begin{array}{l} f_1\left(x_1,x_2,\ldots,x_n\right)=0 \\ f_2\left(x_1,x_2,\ldots,x_n\right)=0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ f_n\left(x_1,x_2,\ldots,x_n\right)=0 \end{array}\right. \qquad (1)$$

If we have a k-position.

$x^k = \begin{bmatrix} x_1^k \\ x_2^k \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ x_n^k \end{bmatrix}$ and the the solution is $x^r = \begin{bmatrix} x_1^r \\ x_2^r \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ x_n^r \end{bmatrix}$. The method Newton counts the next root approach according to form $x^{k+1}=x^k-J\left(x^k\right)^{-1}\cdot F_n\left(x^k\right), k=0,1,2\ldots,$ where **J** Jacobian matrix and x_0 a initial value of approach of the solution.

I.2. Production of the method

For the analytical construction of the Newton - Raphson method, we use Taylor polynomial. The function F_n is deemed a continuous and derivative function. At least: a k-potition and k + 1-potition of approach of the system solution will be for $x^k = \begin{bmatrix} x_1^k \\ x_2^k \\ \cdot \\ \cdot \\ \cdot \\ x_n^k \end{bmatrix}$ of the solution k potition and $x^{k+1} = \begin{bmatrix} x_1^{k+1} \\ x_2^{k+1} \\ \cdot \\ \cdot \\ \cdot \\ x_n^{k+1} \end{bmatrix}$ of the solution $k+1$ potition, which will be the next approximate solution.

To start the proof must be put the meaning of correction and for this process we will have to follow a prosthetic process, so we have the form.

$$\begin{bmatrix} x_1^{k+1} \\ x_2^{k+1} \\ \cdot \\ \cdot \\ \cdot \\ x_n^{k+1} \end{bmatrix} = \begin{bmatrix} x_1^k \\ x_2^k \\ \cdot \\ \cdot \\ \cdot \\ x_n^k \end{bmatrix} + \begin{bmatrix} s_1^k \\ s_2^k \\ \cdot \\ \cdot \\ \cdot \\ s_n^k \end{bmatrix} \text{ where } \begin{bmatrix} s_1^k \\ s_2^k \\ \cdot \\ \cdot \\ \cdot \\ s_n^k \end{bmatrix} \text{ is correction of the approximate solution.} \qquad (2)$$

from relations (1 & 2) arise.

$$\left\langle \begin{array}{l} f_1(x^{k+1}) = f_1(x^k + s^k) = 0 \\ f_2(x^{k+1}) = f_2(x^k + s^k) = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ f_n(x^{k+1}) = f_n(x^k + s^k) = 0 \end{array} \right. \qquad (3)$$

Developing each component with Taylor's type and in order to reverse the second-order terms and above, the following equations arise:

$$\begin{array}{l} f_1\left(x_1^{k+1}, \ldots, x_n^{k+1}\right) = f_1\left(x_1^k, \ldots, x_n^k\right) + s_1^k \dfrac{\partial f_1\left(x_1^k, \ldots, x_n^k\right)}{\partial x_1} + \ldots + s_n^k \dfrac{\partial f_1\left(x_1^k, \ldots, x_n^k\right)}{\partial x_n} \\ f_2\left(x_1^{k+1}, \ldots, x_n^{k+1}\right) = f_2\left(x_1^k, \ldots, x_n^k\right) + s_1^k \dfrac{\partial f_2\left(x_1^k, \ldots, x_n^k\right)}{\partial x_1} + \ldots + s_n^k \dfrac{\partial f_2\left(x_1^k, \ldots, x_n^k\right)}{\partial x_n} \\ \ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots \\ f_n\left(x_1^{k+1}, \ldots, x_n^{k+1}\right) = f_n\left(x_1^k, \ldots, x_n^k\right) + s_1^k \dfrac{\partial f_n\left(x_1^k, \ldots, x_n^k\right)}{\partial x_1} + \ldots + s_n^k \dfrac{\partial f_n\left(x_1^k, \ldots, x_n^k\right)}{\partial x_n} \end{array} \qquad (4)$$

According to Relation (3) we have

$$\begin{array}{l} f_1\left(x_1^{k+1}, \ldots, x_n^{k+1}\right) = f_1\left(x_1^k, \ldots, x_n^k\right) + s_1^k \dfrac{\partial f_1\left(x_1^k, \ldots, x_n^k\right)}{\partial x_1} + \ldots + s_n^k \dfrac{\partial f_1\left(x_1^k, \ldots, x_n^k\right)}{\partial x_n} = 0 \\ f_2\left(x_1^{k+1}, \ldots, x_n^{k+1}\right) = f_2\left(x_1^k, \ldots, x_n^k\right) + s_1^k \dfrac{\partial f_2\left(x_1^k, \ldots, x_n^k\right)}{\partial x_1} + \ldots + s_n^k \dfrac{\partial f_2\left(x_1^k, \ldots, x_n^k\right)}{\partial x_n} = 0 \\ \ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots \\ f_n\left(x_1^{k+1}, \ldots, x_n^{k+1}\right) = f_n\left(x_1^k, \ldots, x_n^k\right) + s_1^k \dfrac{\partial f_n\left(x_1^k, \ldots, x_n^k\right)}{\partial x_1} + \ldots + s_n^k \dfrac{\partial f_n\left(x_1^k, \ldots, x_n^k\right)}{\partial x_n} = 0 \end{array} \qquad (5)$$

The above system can be written in the following format:

$J\left(x^k\right) \cdot s^k = -F_n\left(x^k\right), k = 0, 1, 2 \ldots (6)$ After we define 2 elementary matrices $J\left(x^k\right), F_n\left(x^k\right)$ (7)

So we can give the ultimate relations to the individual solutions from the form $x_i^{k+1} = x_i^k - \det\left(J\left(x^k\right)^{-1}\right) \cdot \det\left(\delta_i\left(F_n\left(x^k\right)\right), k = 0, 1, 2 \ldots\right.$ & $i = 1, 2, \ldots n$ & $J\left(x^k\right) \neq 0$

$x_i^{k+1} = x_i^k - \det\left(J\left(x^k\right)^{-1}\right) \cdot \det\left(\delta_i\left(F_n\left(x^k\right)\right) \Rightarrow\right.$

$$x_i^{k+1} = x_i^k \frac{\begin{vmatrix} \frac{\partial f_1}{\partial x_1} \frac{\partial f_1}{\partial x_2} \dots - f_1\left(x_1^k, x_2^k, \dots, x_n^k\right) \dots \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} \frac{\partial f_2}{\partial x_2} \dots - f_2\left(x_1^k, x_2^k, \dots, x_n^k\right) \dots \frac{\partial f_2}{\partial x_n} \\ \cdots \cdots \quad \cdots\cdots\cdots\cdots\cdots\cdots \quad \cdots \\ \cdots \cdots \quad \cdots\cdots\cdots\cdots\cdots\cdots \quad \cdots \\ \cdots \cdots \quad \cdots\cdots\cdots\cdots\cdots\cdots \quad \cdots \\ \frac{\partial f_n}{\partial x_1} \frac{\partial f_n}{\partial x_2} \dots - f_n\left(x_1^k, x_2^k, \dots, x_n^k\right) \dots \frac{\partial f_n}{\partial x_n} \end{vmatrix}}{\begin{vmatrix} \frac{\partial f_1}{\partial x_1} \frac{\partial f_1}{\partial x_2} \dots \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} \frac{\partial f_2}{\partial x_2} \dots \frac{\partial f_2}{\partial x_n} \\ \cdots \cdots \quad \cdots \\ \cdots \cdots \quad \cdots \\ \cdots \cdots \quad \cdots \\ \frac{\partial f_n}{\partial x_1} \frac{\partial f_n}{\partial x_2} \dots \frac{\partial f_n}{\partial x_n} \end{vmatrix}}, k = 0, 1, 2 \dots, n \tag{8}$$

and $i = 1, 2, \dots, n$, $i =$ potition for $F(x^k)$

According to the formula (8) we calculate all the n roots (where $i = 1, 2, ..n$), with $k + 1$ repetitions.

In order to use this method, however, we must rely on the generalized root finding theorem, which separates each function by categorification.

I.3. Theorem 1.Generalized Theorem of existence and of finding Roots to a Equation.($\mathbf{GR_TE}$)

We assume that we have general, elementary functions (in general transcendental) that are analytic except for some isolated singularities and branch cuts, in which case these and their local inversions will have convergent Taylor series expansions on suitable disks. Under these conditions i consider it a transcendental $f : C \to C$ where $f(x) = \sum\limits_{i=1}^{n} a_i \sigma_i(x) + a_0, a_i, a_0 \in C$ with $\sigma_i(x)$ **functional factors** where $\sigma_i : C \to C$ for which there is likewise an inversion and if suppose G the field of the roots of equation $f(x) = 0$, that defined from relation by category of functional factors σ_i:

$$G = F\left(G_1 = \left\{r_1^{\sigma_1}, r_2^{\sigma_1}, \dots, r_{m_1}^{\sigma_1}\right\}, G_2 = \left\{r_1^{\sigma_2}, r_2^{\sigma_2}, \dots, r_{m_2}^{\sigma_2}\right\}, \dots, G_k = \left\{r_1^{\sigma_k}, r_2^{\sigma_k}, \dots, r_{m_k}^{\sigma_k}\right\}\right) \dots, \\ G_n = \left\{r_1^{\sigma_n}, r_2^{\sigma_n}, \dots, r_{m_n}^{\sigma_n}\right\}\big)$$

with $1 \leq k \leq n$ and $0 \leq m_k \leq \infty$ where G_i **subfields** of total field of roots where produced the σ_i, and G it will be at the same time and the total global field, which belong the roots of the equation $f(x) = 0$.

If we define the relation of function and divide it into 2 parts ie.

$$\sigma_k(x) = u_k = \frac{1}{a_k}\left(-\sum_{i=1, i \neq k}^{n} a_i \sigma_i(x) - a_0\right)$$

then the subfields of roots that produces the $\sigma_k(x)$, $1 \leq k \leq n$ (according to the generalized theorem by Lagrnage method and the (part#1(pag.1) that we mentioned [1]), and they will correspond to G_k. Now if we accept all the functions $\sigma_k(x)$ with $1 \leq k \leq n$ and exist inverse functions (invertible procedure) i.e are 1-1, the equation $f(x) = 0$ is fully resolved, and is represented by the field G. The counting of Roots of equation will be given from the relation $|G| = |G_1 \cup G_2 \cup \ldots \cup G_n|$ where $(\mathrm{G_1}, \mathrm{G_2}, \ldots \mathrm{G_n})$ the individual ones subfields) and will be a bounded integer or an infinite integer.

I.4. Applicable Example: Let's get the system.

$$\left\langle \begin{array}{l} e^x + 2y^2 = 3 \\ x^2 + x \cdot y + y = 1 \end{array} \right.$$

The analysis leads to 6 categories that you need to make sure to categorize per case.

1st Category - Complex Roots Multiple.

We apply the exponential categorization that leads to radical multiplicity as the below form,

$$\left\langle \begin{array}{l} e^x = u \Rightarrow x = \log[u] + 2k\pi I \\ y^2 = v \Rightarrow y = \pm\sqrt{v} \end{array} \right.$$

By replacing the above relation we have the following program in Mathematica,

i) Case 1. We have 1 form depending on the basic categorization reported.

$$\left\langle \begin{array}{l} e^x = u \Rightarrow x = \log[u] + 2k\pi I \\ y^2 = v \Rightarrow y = -\sqrt{v} \end{array} \right.$$

Program in mathematica 1.

```
ClearAll;
Off[FindRoot::jsing,FindRoot::lstol,General::ovfl,FindRoot::sszero,FindRoot::nlnum,General::munfl,
FindRoot::cvmit,General::stop]
For [k = −50, k < 50, k + +,
f1 [u_, v_] := (log[u] + 2*k*I*π)^ 2 − (log[u] + 2*k*I*π)* v^(1/2) − 1;
f2 [u_, v_] := u + 2*v − 3;


h1 [u_, v_] := D[f1[u, v], u];
h2[u_, v_] := D[f2⌈u, v⌉, u⌉;
h3 [u_, v_] := D[f1[u, v], v];
h4 [u_, v_] := D[f2[u, v], v];
u0 = 1/10;
v0 = 1/10;


u1 =u0 − Det[({{−N[f1[u, v]], h3[u, v]}, {−N[f2[u, v]], h4[u, v]}})]/ Det[({{h1[u, v], h3 [u, v]},
{h2[u, v], h4[u, v]}})]/{u− > u0, v− > v0};
v1 = v0 − Det[({{N[h1[u, v]], −f1[u, v]}, {N[h2[u, v]], −f2[u, v]}})]/ Det[({{h1[u, v], h3[u, v]},
{h2[u, v], h4[u, v]}})]/{u − u0, v− > v0};
```

u0 = u1;
v0 = v1;
t1 = N⌈y | /.FindRoot [f1[y, z] == 0&&f2⌈y, z] == 0, {y, u1}, {z, v1}, WorkingPrecision→ 150];
t2 = N[z]/ FindRoot [f1[y, z] == 0&&f2[y, z] == 0, {y, u1}, {z, v1}, WorkingPrecision→ 150];
s1 = (log[t1] + 2*k*I*π);
$s_2 = -\sqrt{t_2}$;
If Abs[f1[t1, t2]] <= 0.0000000001&& Abs[f2[t1, t2]] < 0.0000000001,Goto[mama],Goto[end]]
Label[mama];Print["x(",k,")=",N[s1,30],",","y=",N[s2,30],",","approach =",f1[t1,t2],",",f2[t1,t2]];
Label[end];Loopback]

Results 1. For only 21 roots with ascending order of $\{x, y\}$ & $s_2 = -\sqrt{t_2}$.

$x(-10) = 8.9824864689405 - 62.578197669312\ \mathrm{I},\ y = -7.982486468940525908 + 62.57819766931224256\ \mathrm{I}$
$x(-9) = 8.7729687951075 - 56.2742777431092\ \mathrm{I},\ y = -7.772968795107528080 + 56.274277743109172352\ \mathrm{I}$
$x(-8) = 8.5389096638108 - 49.9661540976586\ \mathrm{I},\ y = -7.538909663810810091 + 49.966154097658598783\ \mathrm{I}$
$x(-7) = 8.2737713192468 - 43.6523199791640\ \mathrm{I},\ y = -7.273771319246788582 + 43.652319979164027581\ \mathrm{I}$
$x(-6) = 7.9679815740151 - 37.3304199286424\ \mathrm{I},\ y = -6.967981574015143709 + 37.330419928642354545\ \mathrm{I}$
$x(-5) = 7.6066889578280 - 30.9965339108696\ \mathrm{I},\ y = -6.606688957827978835 + 30.996533910869577102\ \mathrm{I}$
$x(-4) = 7.1649299300210 - 24.6435677342553\ \mathrm{I},\ y = -6.164929930021001684 + 24.643567734255263124\ \mathrm{I}$
$x(-3) = 6.5954330754842 - 18.2570703447316\ \mathrm{I},\ y = -5.595433075484243467 + 18.257070344731583661\ \mathrm{I}$
$x(-2) = 5.78870297338997 - 11.801849564751\ \mathrm{I},\ y = -4.788702973389965150\!2 + 11.80184956475199277\ \mathrm{I}$
$x(-1) = 4.34535927492546 - 5.1700611138708\ \mathrm{I},\ y = -3.3453592749254563755 + 5.170061113870807241\ \mathrm{I}$
$x(0) = -1.000000000000000 + 0.\ \mathrm{I},\ y = -1.14719670476090491567 + 0.\ \mathrm{I}$
$x(1) = 4.34535927492546 + 5.1700611138708\ \mathrm{I},\ y = -3.3453592749254563755 - 5.170061113870807244\ \mathrm{I}$
$x(2) = 5.78870297338997 + 11.801849564751\ \mathrm{I},\ y = -4.7887029733899651502 - 11.80184956475199271\ \mathrm{I}$
$x(3) = 6.5954330754842 + 18.2570703447316\ \mathrm{I},\ y = -5.595433075484243467 - 18.257070344731583661\ \mathrm{I}$
$x(4) = 7.1649299300210 + 24.6435677342553\ \mathrm{I},\ y = -6.164929930021001684 - 24.643567734255263124\ \mathrm{I}$
$x(5) = 7.6066889578280 + 30.9965339108696\ \mathrm{I},\ y = -6.606688957827978835 - 30.996533910869577102\ \mathrm{I}$
$x(6) = 7.9679815740151 + 37.3304199286424\ \mathrm{I},\ y = -6.967981574015143709 - 37.330419928642354545\ \mathrm{I}$
$x(7) = 8.2737713192468 + 43.6523199791640\ \mathrm{I},\ y = -7.273771319246788582 - 43.652319979164027581\ \mathrm{I}$
$x(8) = 8.5389096638108 + 49.9661540976586\ \mathrm{I},\ y = -7.538909663810810091 - 49.966154097658598783\ \mathrm{I}$
$x(9) = 8.7729687951075 + 56.2742777431092\ \mathrm{I},\ y = -7.772968795107528080 - 56.274277743109172352\ \mathrm{I}$
$x(10) = 8.9824864689405 + 62.5781976693122\ \mathrm{I},\ y = -7.982486468940525908 - 62.57819766931224256\ \mathrm{I}$

Infinite Roots

Case 2. With same condition and $s_2 = \sqrt{t_2}$

Result 2. For only 21 roots with ascending order of $\{x, y\}$

$x(-10) = 8.9824864689405 - 62.5781976693122\ \mathrm{I},\ y = 7.982486468940525908 - 62.578197669312242562\ \mathrm{I}$
$x(-9) = 8.7729687951075 - 56.2742777431092\ \mathrm{I},\ y = 7.772968795107528080 - 56.274277743109172352\ \mathrm{I}$

$$x(-8) = 8.5389096638108 - 49.9661540976586 \text{ I}, \ y = 7.538909663810810091 - 49.966154097658598783 \text{ I}$$
$$x(-7) = 8.2737713192468 - 43.6523199791640 \text{ I}, \ y = 7.273771319246788582 - 43.652319979164027581 \text{ I}$$
$$x(-6) = 7.9679815740151 - 37.3304199286424 \text{ I}, \ y = 6.967981574015143709 - 37.330419928642354545 \text{ I}$$
$$x(-5) = 7.6066889578280 - 30.9965339108696 \text{ I}, \ y = 6.606688957827978835 - 30.996533910869577102 \text{ I}$$
$$x(-4) = 7.1649299300210 - 24.6435677342553 \text{ I}, \ y = 6.164929930021001684 - 24.643567734255263124 \text{ I}$$
$$x(-3) = 6.5954330754842 - 18.2570703447316 \text{ I}, \ y = 5.595433075484243467 - 18.257070344731583661 \text{ I}$$
$$x(-2) = 5.78870297338997 - 11.801849564751 \text{ I}, \ y = 4.7887029733899651502 - 11.80184956475199277 \text{ I}$$
$$x(-1) = 4.34535927492546 - 5.1700611138708 \text{ I}, \ y = 3.3453592749254563755 - 5.170061113870807241 \text{ I}$$
$$x(0) = -1.000000000000000 + 0. * 10 - 16 \text{ I}, \ y = 1.14719670476090491567 + 0. * 10 - 175 \text{ I}$$
$$x(1) = 4.34535927492546 + 5.1700611138708 \text{ I}, \ y = 3.3453592749254563755 + 5.170061113870807244 \text{ I}$$
$$x(2) = 5.78870297338997 + 11.801849564751 \text{ I}, \ y = 4.7887029733899651502 + 11.80184956475199277 \text{ I}$$
$$x(3) = 6.5954330754842 + 18.2570703447316 \text{ I}, \ y = 5.595433075484243467 + 18.257070344731583661 \text{ I}$$
$$x(4) = 7.1649299300210 + 24.6435677342553 \text{ I}, \ y = 6.164929930021001684 + 24.643567734255263124 \text{ I}$$
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$$x(6) = 7.9679815740151 + 37.3304199286424 \text{ I}, \ y = 6.967981574015143709 + 37.330419928642354545 \text{ I}$$
$$x(7) = 8.2737713192468 + 43.6523199791640 \text{ I}, \ y = 7.273771319246788582 + 43.652319979164027581 \text{ I}$$
$$x(8) = 8.5389096638108 + 49.9661540976586 \text{ I}, \ y = 7.538909663810810091 + 49.966154097658598783 \text{ I}$$
$$x(9) = 8.7729687951075 + 56.2742777431092 \text{ I}, \ y = 7.772968795107528080 + 56.274277743109172352 \text{ I}$$
$$x(10) = 8.9824864689405 + 62.578197669312 \text{ I}, \ y = 7.982486468940525908 + 62.57819766931224256 \text{ I}$$

Infinite Roots

Also the another case:

$$\left\langle \begin{array}{l} e^x = u \Rightarrow x = \log[u] + 2k\pi I \\ y^2 = v \Rightarrow y = -\sqrt{v} \end{array} \right.$$

Result.3 Gives the 2 couples really the we estimated before.

x= -1.000000000000000, y= 1.14719670476090491567
x= -1.000000000000000, y= -1.14719670476090491567

2st Category - Reals Roots.

We apply polynomial categorization that leads to 4 cases that arise after 4 combinations of signs, as the below form,

$$\left\langle \begin{array}{l} x^2 = u \Rightarrow x = \pm\sqrt{u} \\ y^2 = v \Rightarrow y = \pm\sqrt{v} \end{array} \right.$$

By replacing the above relation we have the following program in Mathematica,

*Case 1. We have 1 form depending on the basic categorization reported.

$$\left\langle \begin{array}{l} x^2 = u \Rightarrow x = +\sqrt{u} \\ y^2 = v \Rightarrow y = +\sqrt{v} \end{array} \right.$$

Program in mathematica 2.

```
ClearAll;
For[k = −1, k < 1, k + +,
f1[u_, v_] := (u) + (u)^(1/2)*(v)^(1/2) + (v)^(1/2) − 1;
f2 [u_, v_] := Exp [u^(1/2)]* + 2*v^(1/2) − 3;

h1[u_,v_]  := D[u[u, v], u];
h2[u_,v_]  := D[f2[u, v], u];
h3 [u_, v_] := D[f1[u, v], v];
h4 [u_, v_] := D[f2[u, v], v];
u0 = −1/10;
v0 = −1/10;
For[m = 1,  m < 2,  m + +,
u1 = u0 − Det[({{−N[f1[u, v]], h3[u, v]}, {−N[f2[u, v]], h4[u, v]}})]/ Det[({{h1[u, v], h3[u, v]},
{h2[u, v], h4[u, v]}})]/.{u− > u0, v− > v0};
v1=v0-Det [({{N[h1[u, v]], −f1[u, v]}, {N[h2[u, v]], −f2[u, v]}})]/ Det[({{h1[u, v], h3[u, v]},
{h2[u, v], h4[u, v]}})]/{u− > u0, v− > v0};
u0 = u1;
v0 = v1;
t1 = N[y]/. FindRoot[f1[y, z] == 0&&f2[y, z] == 0, {y, u1}, {z, v1},WorkingPrecision − > 150];
t2 = N[z]/.FindRoot[f1[y, z] = 0&&f2[y, z] = 0, {y, u1}, {z, v1},WorkingPrecision − > 150];
s1 = √t1
s2 = √t2;
If Abs[f1[t1, t2]] <= 0.0000000001 && Abs[f2[t1, t2]] < 0.0000000001,Goto[mama],Goto[end]];
Label[mama];
Print[”x(”, k, ”) = ”, N[s1, 30], ”, ”, ”y = ”, N[s2, 30], ”, ”,” approach = ”, f1[t1, t2], ”, ”, f[t1, t2]]; Label[end];Loopback]]
```

Result4. For only 1 root for any value of $\{x, y\}$.

$x = 0.00000000000, y = 1.000000000000$

Case 2. We have 1 form depending on the basic categorization reported.

$$\left\langle \begin{array}{l} x^2 = u \Rightarrow x = +\sqrt{u} \\ y = v \Rightarrow y = +v \end{array}\right.$$

Program in mathematica 3.

```
ClearAll;
For [k = −5, k < 5, k + +,
f1[u_, v_] := (u) + (u)^(1/2)*(v) + (v) − 1;
f2 [u_, v_] := Exp [u^(1/2)] + 2*v^2 − 3;

h1[u_, v_] := D[f1[u, v], u];
h2[u_, v_] := D[f2[u, v], u];
h3[u_, v_] := D[f1[u, v], v];
h4[u_, v_] := D[f2[u, v], v];
u0 = −1/10;
v0 = −1/10;
For[m = 1, m < 2, m + +,
u1 = u0 − Det[({
{−N[f1[u, v]], h3[u, v]},
{−N[f2[u, v]], h4[u, v]}})]/ Det[({{h1[u, v], h3[u, v]}, {h2[u, v], h4[u, v]}})]/.{u− > u0, v− > v0};
v1=v0-Det [({{N[h1[u, v]], −f1[u, v]}, {N[h2[u, v]], −f2[u, v]}})]/ Det[({{h1[u, v], h3[u, v]}, {h2[u, v], h4[u, v]}})]/.
{u− > u0, v− > v0};
u0 = u1;
v0 = v1;
t1 = N[y]/. FindRoot[f1[y, z] = 0&&f2[y, z] == 0, {y, u1}, {z, v1},WorkingPrecision− > 150];
t2 = N[z]/.FindRoot[f1[y, z] == 0&&f2[y, z] = 0, {y, u1}, {z, v1},WorkingPrecision− > 150];
s1 = √t1.
s2 = t2;
If[Abs[f1[t1, t2]] <= 0.0000000001&& Abs[f2[t1, t2]] < 0.0000000001, Goto[mama],Goto[end]];
Label[mama];
Print [”x(”, k, ”) = ”, N[s1, 30], ”, ”, ”y = ”, N[ s2, 30], ”, ” , ”approach = ”, f1[t1, t2], ”, ”,f2[t1, t2]] ;
Label[ end];Loopback]]
```

Result 5. For only 1 root for any value of $\{x, y\}$.

x=1.09284848265209, y=-0.0928484826520924

II. Method of Generalized Theorem of existence and of finding roots to a equation.(GRIM).

A method that uses the generalized theorem we encountered for solutions of an equation of a variable. It is the same powerful by finding a radical of a variable, simply here we **use relaxation levels**, which facilitate the detection of roots. Is a similar program, but adapted to the requirements of a system.

This problem is defined as follows:

Is given $G_n = (g_1, g_2, \ldots, g_n) : D_n \subset C^n \to C^n$ to be found $x^r \in D_n : G_n(x_n) = \theta_n$ where $\theta_n = (0, 0, .., 0)^T$ and where $g_i(x), i = 1, 2 \ldots n$ for $x = (x_1, x_2, .., x_n)^T \in C^n$ denotes the i-position Component of G_n. The equation $G_n(x_n) = \theta_n$ written,

$$G_n(x_n) = \left\langle \begin{array}{l} g_1(x_1, x_2, \ldots, x_n) = 0 \\ g_2(x_1, x_2, \ldots, x_n) = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ g_n(x_1, x_2, \ldots, x_n) = 0 \end{array} \right. \tag{1}$$

II.1. Definitions for general form and we make 4 basic processes

II.1.1. We define the a more generalised multivariate functional equations as follows,

$$g_r\left(x_q, x_{q+1}, \ldots, x_m\right) = \sum_{l=1}^{n'} a_l \cdot \prod_{i=1}^{k} f_{l_i}\left(\prod_{j=q}^{m} x_j\right), 1 \leq n', q, m, k \leq n, \{r, m, l, k, n, n'\} \in N^+ \tag{2}$$

where $i = 1, 2, \ldots, k$ & $1 \leq j \leq m,\ 1 \leq r \leq n',\ m \leq n$.

II.1.2. The case $g_r\left(x_1, x_2, \ldots, x_{n'}\right) = \sum\limits_{q=1}^{k} a_q \cdot f_q\left(x_q\right),\ 1 \leq r, k,\ n' \leq n, \{q, n, n'\} \in N^+$

We do the complementary multivariate functions of $G_n = (g_1, g_2, \ldots, g_n)$ the $h_2g_1\left(x_2, x_3 \ldots, x_n\right)$, $h_2g_2\left(x_1, x_3 \ldots, x_n\right), \ldots, h_2g_n\left(x_1, x_3 \ldots, x_{n-1}\right)$ **that does not contain the variable we want to calculate**. Each complementary function will be equal to the relation that given below:

$$\begin{aligned}
&h_2g_1\left(x_2, x_3 \ldots, x_n\right) = -1/a_1 \cdot \sum_{q>1}^{n} a_q \cdot f_q\left(x_q\right), n, q \in N^+. \\
&h_2g_2\left(x_1, x_3, \ldots, x_n\right) = -1/a_2 \cdot \sum_{q=1, q\neq 2}^{n} a_q \cdot f_q\left(x_q\right), n, q \in N^+. \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&h_2g_n\left(x_1, x_2, \ldots, x_{n-1}\right) = -1/a_n \cdot \sum_{q=1}^{n-1} a_q \cdot f_q\left(x_q\right), n, q \in N^+.
\end{aligned} \tag{3}$$

II.1.3. We find the inverse functions $h_2g_r(x_1, x_2, \ldots, x_{r-1}, x_{r+1}, \ldots, x_n) = f_r(x_r) = u \Leftrightarrow x_r = f_r^{-1}(u)$ of $f_r(x_r)$, $1 \leq r \leq n$, for every variable x_r. This is a process we made with generalized solution of the equation of one variable.

II.1.4. We define the repetitive process that will have for any variable x_r (if c are initial values and l_r repetitions). But making reversal to the function we come to the relation, $x_r = f_r^{-1}\left(f\left(x_r\right)\right)$ and that looks in the relationship below and will be equal to the form

$$x_r = \text{Nest}\left(f_r^{-1}\left(h_2g_r(\#, \#, \ldots, \#)\&, c_r, l_r\right), \quad 1 \leq r \leq n.\right. \tag{4}$$

In order to complete the procedure, relation (4), which must be related to (3), includes one or more integer multiplicity coefficients of complex numbers or trigonometric in the case of categorisation. Each coefficient is also involved in the equations of special functions.

The previous definitions can only see the real dimension if the following theorem of the generalized method (GRIM) for solving roots of polynomial or transcendental equations holds.

II.1.5. Theorem 2. Generalized Iterative Method approximation of Roots (GRIM) [3]

We assume that we have general, elementary functions(in general transcendental) that are analytic except for some isolated singularities and branch cuts, in which case these and their local inversions will have convergent Taylor series expansions on suitable disks. Under these conditions for the determination of the roots any equation $f(x) = \sum\limits_{i=1}^{n} a_i\sigma_i(x) - a_0 = 0,\ \{x, a_i, a_0 \in \mathbb{C}, i \in \mathbb{N}^+\}$ (1) where $f : C \to C$ which consists from distinct monomials functions $\sigma_i(x), 1 \leq i \leq n \wedge a_i \in C$ and for which there is likewise an inversion, after we accept $\sigma_0(x) = 1$, exist a repeating relation for every subfield of roots $G_l, G_2, \ldots, G_r, \ldots G_n$ and k the counting of the roots for any $G_r, r \in \{1, 2, \ldots, n\}$. Now if we now assume that that x_0 initial value for x then and from $\mathrm{GR_TE}$ it will hold more closely with indices and for the any roots of f(x) will apply:

$$\begin{aligned} i)\ {}^{m_r}_{\sigma_r}x^{\mu}_{k_r} &= \sigma_r^{-1}\bigg(\frac{\alpha_0}{\alpha_r} - \frac{1}{\alpha_r}\sigma_r^c\bigg(\sigma_r^{-1}\bigg(\frac{\alpha_0}{\alpha_r} - \frac{1}{\alpha_r}\sigma_r^c\bigg(\sigma_r^{-1}\bigg(\frac{\alpha_0}{\alpha_r} - \ldots - \\ &\quad -\frac{1}{\alpha_r}\sigma_r^c\Bigg(\sigma_r^{-1}\Bigg(\frac{\alpha_0}{\alpha_r} - \frac{1}{\alpha_r}\cdot\sum_{\mathrm{i=1,i\neq r}}^{\mathrm{n}}\alpha_i\sigma_{\mathrm{i}}\left(\mathrm{x}_0\right)\Bigg)\Bigg)\Bigg)\Bigg)\Bigg)\Bigg)\Bigg), 1 < \mathrm{r} \leq \mathrm{n}, 1 \leq \mathrm{k} \leq \infty \end{aligned}$$

$$\begin{aligned} ii)\ {}^{m_r}_{\sigma_r}x^{\mu}_{k_r} &= \sigma_1^{-1}\bigg(\frac{\alpha_0}{\alpha_1} - \frac{1}{\alpha_1}\sigma_1^c\bigg(\sigma_1^{-1}\bigg(\frac{\alpha_0}{\alpha_1} - \frac{1}{\alpha_1}\sigma_1^c\bigg(\sigma_1^{-1}\bigg(\frac{\alpha_0}{\alpha_1} - \ldots - \frac{1}{\alpha_1}\sigma_1^c\bigg(\sigma_1^{-1}\bigg(\frac{\alpha_0}{\alpha_1} - \\ &\quad -\frac{1}{\alpha_1}\cdot\sum_{\mathrm{i=2}}^{\mathrm{n}}\alpha_i\sigma_{\mathrm{i}}\left(\mathrm{x}_0\right)\Bigg)\Bigg)\Bigg)\Bigg)\Bigg)\Bigg)\Bigg), \mathrm{r} = 1, 1 \leq \mathrm{k} \leq \infty \end{aligned}$$

where σ_r^c the complement of σ_r.

where μ the number of repetitions and is based on the iterative procedure of generalized theorem $\mathrm{GR_TE}$. This relation is also the generalized recurrent periodically algorithm of finding of the roots of any equation.

Proof

For proof we go from the special to the general case. Suppose we have the function $f[x] = \sum_{i=1}^{\infty}\alpha_i\sigma_i(x) - \alpha_0 = 0$ (1) where $f : C \to C, \alpha_i \in C$ where α_i constants. I separate the initial equation with a sum of functions and a monomial function and then therefore the two parts they will have sum equal to zero. This means as example: $f_1(x) + f_2(x) - \alpha_0 = 0$ where: $f_1(x) = a_1\sigma_1(x) \wedge f_2(x) = \sum_{i=2}^{n} a_i\sigma_i(x) \Rightarrow \alpha_1\sigma_1(x) = \alpha_0 - f_2(x) \Rightarrow$ ${}_{\sigma_1}x_{j+1} = \frac{1}{\alpha_1}\cdot\sigma_1^{-1}\left(\alpha_0 - f_2(x_j)\right)$ (3) which results from the inverse of the first function with the simultaneous repetitive procedure. The result gives one of roots of subfield G_1.

If where x_0 is initial value from x and which results after from μ repetitions, take into account the same and the other fields $G_1, G_2, \ldots, G_r, \ldots G_n$. If also I call $\sigma_{\mathrm{r}}^{\mathrm{c}}(\mathrm{x}) = \sum_{\mathrm{i=1,i\neq r}}^{\mathrm{n}}\alpha_{\mathrm{i}}\cdot\sigma_{\mathrm{i}}(\mathrm{x})$ (4) is complement of $\sigma_{\mathrm{r}}(\mathrm{x})$ where x_0 initial value for x then and from $\mathrm{GR_TE}$ (Theorem 1) will be apply for the any roots of $f(x)$:

from (1)

$$\begin{aligned} \alpha_1\sigma_1(x) = -\left(\sum_{i=2}^{n}\alpha_i\sigma_i(x) - \alpha_0\right) &\Leftrightarrow \sigma_1(x) = -\left(\frac{1}{\alpha_1}\sum_{i=2}^{n}\alpha_i\sigma_i(x) - \alpha_0\right) \\ &\Leftrightarrow x = \sigma_1^{-1}\left(-\left(\frac{1}{\alpha_1}\sum_{i=2}^{n}\alpha_i\sigma_i(x) - \alpha_0\right)\right) \qquad (5) \end{aligned}$$

and then apply for repeat with simple procedure ${}_{\sigma_r}x_{j+1} = \sigma_r^{-1}\left(-\frac{1}{a_r}\left(\sum_{i\neq r, i=1}^{n}\alpha_i\cdot\sigma_i\left(x_j\right) - \alpha_0\right)\right)$ (6)

We can now easily deduce the generalized relationship

$$\begin{aligned} i)\ {}^{L_{k,q}}_{\sigma_k} x^{\mu}_{s_q} &= \sigma_k^{-1}\Bigg(\frac{\alpha_0}{\alpha_k}-\frac{1}{\alpha_k}\sigma_k^c\Bigg(\sigma_k^{-1}\Bigg(\frac{\alpha_0}{\alpha_k}-\frac{1}{\alpha_k}\sigma_k^c\Bigg(\sigma_k^{-1}\Bigg(\frac{\alpha_0}{\alpha_k}-\ldots- \\ &- \frac{1}{\alpha_k}\sigma_k^c\Bigg(\sigma_k^{-1}\Bigg(\frac{\alpha_0}{\alpha_k}-\frac{1}{\alpha_k}\cdot\sum_{i=1,i\neq k}^{n}\alpha_i\sigma_i\left(x_0\right)\Bigg)\Bigg)\Bigg)\Bigg)\Bigg)\Bigg)\Bigg)\Bigg) \\ &\text{if } 1<\mathrm{k}\leq \mathrm{n}, 0\leq \mathrm{s}\leq\infty, \mathrm{q}\in \mathrm{N}^+, \end{aligned}$$

$$\begin{aligned} ii)\ {}^{L_{k,q}}_{\sigma_k} x^{\mu}_{s_q} &= \sigma_1^{-1}\Bigg(\frac{\alpha_0}{\alpha_1}-\frac{1}{\alpha_1}\sigma_1^c\Bigg(\sigma_1^{-1}\Bigg(\frac{\alpha_0}{\alpha_1}-\frac{1}{\alpha_1}\sigma_1^c\Bigg(\sigma_1^{-1}\Bigg(\frac{\alpha_0}{\alpha_1}-\ldots- \\ &- \frac{1}{\alpha_1}\sigma_1^c\Bigg(\sigma_1^{-1}\Bigg(\frac{\alpha_0}{\alpha_1}-\frac{1}{\alpha_1}\cdot\sum_{i=2}^{n}\alpha_i\sigma_i\left(x_0\right)\Bigg)\Bigg)\Bigg)\Bigg)\Bigg)\Bigg)\Bigg)\Bigg) \\ &\text{if } \mathrm{k}=1, 0\leq \mathrm{s}\leq\infty, \mathrm{q}\in \mathrm{N}^+ \end{aligned}$$

$$\text{Apply}: \alpha_k\sigma_k(x)+\sigma_k^c(x)-\alpha_0=0\wedge\sigma_k^c(x)=\sum_{i=1,i\neq k}^{n}\alpha_i\sigma_i(x) \text{ where } \sigma_{\mathrm{k}}^{\mathrm{c}} \text{ the complement of } \sigma_k.$$

where σ_r^c the complement of σ_r. The $\mathrm{L_{k,q}}$ is the subfield concerns form (σk). But the k-subfield itself can have q categories.Also where μ the number of repetitions, m_r emerges from the number produced by categorizing the roots for each $G_r, r\in\{1,2,\ldots,n\}$, field and $\mathrm{x_0}$ initial value for x that is based on the iterative procedure of generalized theorem $\mathrm{GR_TE}$.

II.2. Definition - Categorization. [8, 9, 10, 11, 12]

For each discrete function, after we equating to an independent variable other than the variables i have, *i* do the inverse function of the original variable it contains. Then i get a new function containing the independent variable and a coefficient or coefficients of multiplicity of complex or trigonometric form Each coefficient always takes a number integer positive or negative.

II.2.1 Programming procedures.

1. Choice of correct initial values.

It is not always easy to choose the initial value, but we know 2 regions the interval $(-1,1)$ and we find the maximum value r of the full function, which is into region $[1, r>>1)$ or $(-r<<1,-1]$ where $r\in\mathrm{Z}^+$

2. Level of relaxation - Tightening on repetitions.

We divide the For-Next procedure of **κ-Multiple** of Roots into a negative and positive interval and in each part we find the corresponding iterations with requirements per groups with ideal initial values. We vary accordingly the number of iterations and the initial values per negative and positive interval of each part to obtain a correct value response $\kappa\in Z^+$.

3. Newton approach class per groups and μ - Multiple filters $\mu\in Z^+$

II.2.2. There are 3 Methods to solve a system with this method GRIM

1. I make **transformation for every variable per function by categorization** per equation, cyclically, and for all system equations and all the variables and after, i make replacing all variables successively.

2. I make **transformation all functions by categorizing per variable**. After, i do replace the variables successively in each equation with the corresponding variables resulting from the inverse procedure, so as to replace all previous variables with new ones.

3. We **Exchange the initial values of variables**, i.e., where $x_{in} = x_0 \wedge y_{in} = y_0$ and thus calculate again the values of the variables for the system. Therefore, I can make the variables symmetrical and to we find other roots that are likely to exist. For n variables we have $n\,!$ cases. **Example:** ($n = 3$ we have $x(yz, zy)$ and $y(xz, zx)$ and $z(xy, yx)$, total $3! = 6$ cases).

The methods will be discussed in detail in the following examples how and where are applied.

III. Applied examples in the solution methods

III.1.Example(System 2X2) : We assume that we have the system,

$$\left\langle \begin{aligned} & e^u + 2v^2 = 3 \\ & u^2 + u \cdot v + v = 1 \end{aligned} \right.$$

The analysis leads to 2 categories and 5 cases, that you need to make sure to categorize per case.

III.2. 1^{st} Category - Complex Roots Multiple (We use Method 2).

We apply the exponential categorization that leads to radical multiplicity as the below form,By replacing the above relation we have the following program in Mathematica,

III.2.1 Case I. We have 1 form depending on the basic categorization reported $v = y$ that like we have the form,

$$\left\langle \begin{aligned} & e^u = x \Rightarrow u = \log[x] + 2k\pi I \\ & v = y \end{aligned} \right.$$

Program in mathematica

```
Clearall;OffGeneral::munfl, FindRoot::cvmit];
h2g1[x_, y_] := N[3 − 2y^2];
h2g2[x_, y_] := N[1 − (Log[x] + 2*k1*I\[Pi])^2 − (Log[x] + 2*k1*I*\[Pi])*y + y − 1];
g1[x_, y_] := N[x + 2*y^2 − 3];
g2[x_, y_] := N[(Log[x] + 2*k1*I*\[Pi])^2 + (Log[x] + 2*k1*I*\[Pi])*y + y − 1];
h1[u_,v_]  := N[Exp[u]+1*v^2 − 3];
h2[u_,v_]  := N[(u^2)+v + u*v − 1];
c := −1/10; d := 10;
For[k1 = −d, k1 <= d, k + +,;
f1[u1_] := N[u1];
f2[u2_] := N[u2];
xr1 = N[Nest[f1[h2g1[#, #]]&, N[c], 4], 100];
```

```
yr2 = N[Nest[f2[h2g2[#, #]]&, N[c], 4], 100];
FQ1 = N[xr1, 15];
FQ2 = N[yr2, 15]; s1 = N[x1, 100]/.
FindRoot[{x1 + 2*y1^2 - 3 == 0, (Log[x1] + 2*k1*I*\[Pi])^2 + (Log[x1] + 2*k1*I*\[Pi])^y1 + y1 - 1 == 0},
{x1, FQ2}, {y1, FQ1}, WorkingPrecision- > 450];
s2 = N[y1, 100]/.
FindRoot[{x1 + 2*y1^2 - 3 == 0, (Log[x1] + 2*k1*I*\[Pi])^2 + (Log[x1] + 2*k1*I*\[Pi])^y1 + y1 - 1 == 0},
{x1, FQ2}, {y1, FQ1}, WorkingPrecision- > 450];
t1 = (Log[s1] + 2*k1*I*/[Pi]); t2 = s2;
Print["(", k1, ")", N[t1, 70], N[t2, 70]];
Print[N[h1[t1, t2], 45], ",", N[h2[t1, t2], 45]]]
```

Result 1. For 11 roots with Infinite Roots of $\{x, y\}$

```
(k1 = -5), x = 7.6066889578280 - 30.9965339108696 I, y = -6.6066889578280 + 30.9965339108696 I
(k1 = -4), x = 7.1649299300210 - 24.6435677342553 I, y = -6.1649299300210 + 24.6435677342553 I
(k1 = -3), x = 6.5954330754842 - 18.2570703447316 I, y = -5.5954330754842 + 18.2570703447316 I
(k1 = -2), x = 5.78870297338997 - 11.801849564751 I, y = -4.78870297338997 + 11.801849564751 I
(k1 = -1), x = 4.34535927492546 - 5.170061113870 I, y = -3.34535927492546 + 5.17006111387081 I
(k1 = 0), x = -1.00000000000000000000000000000, y = -0.0928484826520924769780 94
(k1 = 1), x = 4.34535927492546 + 5.17006111387081 I, y = -3.34535927492546 - 5.170061113870 I
(k1 = 2), x = 5.78870297338997 + 11.801849564751  I, y = -4.78870297338997 - 11.801849564751 I
(k1 = 3), x = 6.5954330754842 + 18.2570703447316 I, y = -5.5954330754842 - 18.2570703447316 I
(k1 = 4), x = 7.1649299300210 + 24.6435677342553 I, y = -6.1649299300210 - 24.6435677342553 I
(k1 = 5), x = 7.6066889578280 + 30.9965339108696 I, y = -6.6066889578280 - 30.9965339108696 I
```

Infinite Roots

As we see, we also have one couple of real roots in position $k1 = 0$ and others are complex roots.

III.2. 2^{nd} Category - Real Roots.

III.2.1. Case I. We have 1 form depending on the basic categorization reported $u = \sqrt{x}, v = -\sqrt{y}$

That like we have the form,

$$\begin{cases} u^2 = x \Rightarrow u = \sqrt{x} \\ v^2 = y \Rightarrow v = -\sqrt{y} \end{cases}$$

Program in mathematica

```
ClearAll;
Off[General::munfl, FindRoot::cvmit];
h2g1 [x_, y_] := N[(3 - Exp[Sqrt[x]])/2];
h2g2[x_, y_] := N [1+ Sqrt [y] + (Sqrt[x])*Sqrt[y]];
g1[x_, y_] := N [Exp[Sqrt[x]] + 2*y - 3] ;
g2[x_, y_] := N [(x) - (Sqrt[x])* Sqrt[y] - Sqrt[y] - 1];
```

```
h1 [u_ + v_] := N [Exp[u] + 2*v^2 − 3];
h2[u_, v_] := N [(u^2) + v + u*v − 1] ;
c := −1/10; d := 1;
For[k1 = −d, kl <= d, k1 + +,
f1[ul_] := N[ul];
f2[u2_] := N[u2];
xrl = N[Nest[f1[h2g1[#, #]]&,  N[c], 4], 100];
yr2 = N[Nest[f2[h2g2[#, #]] &, N[c], 4], 100];
FQ1 = N[xr1, 15];
FQ2 = N[yr2, 15];
s1 = N[x1, 100]/.
FindRoot[{Exp[Sqrt [x1]] + 2*y1 − 3 == 0, (x1) − (Sqrt[x1])* Sqrt[y1] − Sqrt[y1] − 1 == 0} , {x1, FQ1}, {y1, FQ2},
WorkingPrecision− > 450];
s2 = N[yl, 100]/.
FindRoot[{Exp[Sqrt[x1]] + 2*y1 − 3 == 0, (x1) − (Sqrt[x1])*Sqrt[y1] − Sqrt[y1] − 1 == 0}, {x1, FQ1}, {y1, FQ2},
WorkingPrecision− > 450];
t1 = Sqrt[(s1)];
t2 = − Sqrt[s2];
Print[”(”, k1, ”)”, N[t1, 70], N[t2, 70]];
Print[N[h1[t1, t2], 45], ”, ”, N[h2[t1, t2], 45]]]
```

Result 2. For 1 root with Infinite Roots of $\{x, y\}$

(k1 = 0)x = 1.092848482652092476, y = −0.092848482652092476978090

III.2.2. Case II. We have 1 form depending on the basic categorization reported $u = -\sqrt{x}, v = -\sqrt{y}$ hat like we have the form,

$$\left\langle \begin{array}{l} u^2 = x \Rightarrow u = -\sqrt{x} \\ v^2 = y \Rightarrow v = -\sqrt{y} \end{array} \right.$$

Program corresponding in mathematica

```
ClearAll;OffGeneral::munfl, FindRoot::cvmit];
h2g1 [x_y_] := N[(3 − Exp[−Sqrt[x]])/2];
h2g2 [x_y_] := N [1 + Sqrt[y] − (Sqrt[x])*Sqrt[y]];
g1[x_, y_] := N [Exp[− Sqr[x]] + 2*y − 3] ;
g2[x_, y_] := N [(x) + (Sqrt[x])*Sqrt[y] − Sqrt[y] − 1] ;
h1[u_, v_] := N [Exp[u] + 2*v^2 − 3] ;
h2 [u_, v_] := N [(u^2) + v + u*v − 1];
c := −1/10; d := 1;
For [kl = −d, kl <= d, kl + +,
f1[u1_] := N[u1];
f2[u2_] := N[u2];
xrl = N[Nest[f1[h2g1[#, #]]&, N[c], 4], 100];
yr2 = N[Nest[f2[h2g2[#, #]] &,N[c],4],100];
FQ1 = N[xr1, 15];
```

```
FQ2 = N[yr2, 15];
s1 = N[x1, 100]/.FindRoot[{Exp[− Sqrt[x1]] + 2*y1 − 3 == 0, (x1) + (Sqrt[x1])* Sqrt[y1] − Sqrt[y1] − 1 == 0},
{x1, FQ2}, {y1, FQ1},WorkingPrecision − > 450];
s2= N[y1, 100]/.FindRoot[{Exp[−Sqrt[x]] + 2*y1 − 3 == 0, (x1) + (Sqrt[x1])* Sqrt[y1] − Sqrt[y1] − 1 == 0} ,
{x1, FQ1}, {y1, FQ2}, WorkingPrecision− > 450];
t1 = −Sqrt[(s1)];
t2 =−Sqrt[s2];Print["(", k1,")", N[t1, 70], N[t2, 70]];
Print[N[h1[t1, t2], 45],",",N[h2[t1, t2], 45]]]
```

Result 3. For 1 root with Infinite Roots of $\{x, y\}$

$(\mathrm{kl} = 0), \mathrm{x} = -1.000000000000000, \mathrm{y} = -1.14719670476090491566791$

III.2.3. Case III. We have 1 form depending on the basic categorization reported $u = -\sqrt{x}, v = \sqrt{y}$ that like we have the form,

$$\left\langle\begin{array}{l} u^2 = x \Rightarrow u = -\sqrt{x} \\ v^2 = y \Rightarrow v = \sqrt{y} \end{array}\right.$$

```
ClearAll;Off[General::munfl, FindRoot::cvmit];
h2g1 [x_, y_] := N[(3 − Exp[−Sqrt[x]])/2];
h2g2[x_, y_] := N[1−Sqrt[y] + (Sqrt[x])*Sqrt[y]];
g1 [x_, y_] := N[Exp[−Sqrt[x]] + 2*y − 3];
g2[x_, y_] := N[(x) − (Sqrt[x])* Sqrt[y] + Sqrt[y] − 1];
h1 [u_, v_] := N [Exp[u] + 2*v^2 − 3];
h2 [u_, v] := N [(u^2) + v + u*v − 1] ;
c := −1/10; d := 1;
For[k1 = −d, k1 <= d, k1 + +,
f1[u1_] := N[u1];
f2[u2_] := N[u2];
xr1 = N[Nest[f1[h2g1[#, #]]&, N[c], 4], 100];
yr2 = N[Nest[f2[h2g2[#, #]] &,N[c],4],100];
FQ1 = N[xr1, 15];
FQ2 = N[yr2, 15];
s1 = N[x1, 100]/.FindRoot[{Exp[−Sqrt[x1]] + 2*y1 − 3 == 0, (x1) − (Sqrt[x1])* Sqrt[y1] + Sqrt[y1] − 1 == 0},
{x1, FQ2}, {y1, FQ1},WorkingPrecision− > 450];
s2 = N[y1, 100]/.FindRoot [{Exp[−Sqrt[x1]] + 2*y1 − 3 == 0, (x1) − (Sqrt[x1])*Sqrt[y1] + Sqrt[y1] − 1 == 0} ,
{x1, FQ1}, {y1, FQ2},WorkingPrecision− > 450];
t1 = −Sqrt[(s1)];
t2 = Sqrt[s2];
Print["(",k1,")",N[t1, 70],N[t2,70]];
Print[N[h1[t1, t2], 45],",",N[h2[t1, t2],45]]]
```

Result 4. For 1 root with Infinite Roots of $\{x, y\}$

$(\mathrm{k1} = 0), \mathrm{x} = -1.000000000000000, \mathrm{y} = +1.14719670476090491566791$

III.3. We use Method 1. (I do not change variables)

III.3.1. Case I. We have 1 form depending on the basic categorization reported, by complex roots

$x = \mathrm{Log}\,(u_1) + 2 \cdot k \cdot \pi \cdot I, y = u_2$

That like we have the program:

Program in mathematica

```
Off[General::munfl, FindRoot::cvmit];
h2g1 [x_, y_] := N [3 − 2*y^2];
h2g2[x_, y_] := N [(− (x^2) + 1 − y*x)];
g1 [x_, y_] := N [Exp[x] + 2*y^2 − 3];
g2[x_, y_] := N [(x)^2 + y*(x) + y − 1];
h1[u_, v_] := N [Exp[u] + 2*v^2 − 3];
h2[u_, v_] := N [(u^2) + v + u*v − 1]
c := −1/10; d := 10;
For[k1= −d, k1 <= d, k1 + +,
f1 [u1_] := N [Log[u1] + 2*k1*I*\Pi]] ;
f2[u2_] := N[u2];
xr1 = N[Nest[f1[h2 g1[#, #]]& N[c], 4], 100];
yr2=N[Nest[f2[h2g2[#, #]]&, N[c], 4], 100];
FQ1 = N[xr1, 15];
FQ2 = N[yr2, 15];
s1 = N[x1, 100]/.FindRoot[{Exp[x1] + 2*y1^2 − 3 == 0, x1^2 + y1*x1 + y1 − 1 == 0} , {x1, FQ1} ,
{y1, FQ2},WorkingPrecision− >450];
s2 = N[y1, 100]/.FindRoot[{Exp[x1] + 2*y1^2 − 3 == 0, x1^2 + y1*x1 + y1 − 1 == 0} , {x1, FQ1} ,
{y1, FQ2},WorkingPrecision− > 450];
Print[”(”,k1,”)”,N[t1,70],N[t2,70]];
Print[N[g1[t1, t2], 45], ”, N[g2[t1, t2], 45]]]
```

Result 5. For 11 roots with Infinite Roots of {x, y}

```
(k1 = −5), x = 7.6066889578280 − 30.9965339108696 I, y = −6.6066889578280 + 30.9965339108696 I
(k1 = −4), x = 7.1649299300210 − 24.6435677342553 I, y = −6.1649299300210 + 24.6435677342553 I
(k1 = −3), x = 6.5954330754842 − 18.2570703447316 I, y = −5.5954330754842 + 18.2570703447316 I
(k1 = −2), x = 5.78870297338997 − 11.801849564751 I, y = −4.78870297338997 + 11.801849564751 I
(k1 = −1), x = 4.34535927492546 − 5.1700611138708 I, y = −3.34535927492546 + 5.1700611138708 I
(k1 = 0), x = −1.00000000000000000000000000000, y = −0.0928484826520924769780 94
(k1 = 1), x = 4.34535927492546 + 5.1700611138708 I, y = −3.34535927492546 − 5.1700611138708 I
(k1 = 2), x = 5.78870297338997 + 11.801849564751 I, y = −4.78870297338997 − 11.801849564751 I
(k1 = 3), x = 6.5954330754842 + 18.2570703447316 I, y = −5.5954330754842 − 18.2570703447316 I
(k1 = 4), x = 7.1649299300210 + 24.6435677342553 I, y = −6.1649299300210 − 24.6435677342553 I
(k1 = 5), x = 7.6066889578280 + 30.9965339108696 I, y = −6.6066889578280 − 30.9965339108696 I
```

Infinite Roots

III.3.2. Case II. We have 1 form depending on the basic categorization reported,

$$\left\langle\begin{aligned}&x^2 = u \Rightarrow x = +\sqrt{u}\\&y = v\end{aligned}\right.$$

Then we have the corresponding program:

Program in mathematica

```
Off[General:munfl,FindRoot:cvmit];
h2g1 [x_, y_] := N[(3 − Exp[x])/2];
h2g2 [x_, y_] := N [(−x^2 + 1 − y) /y];
g1 [x_, y_] := N [Exp[x] + 2*y^2 − 3];
g2 [x_, y_] := N [(x)^2 + y*(x) + y − 1];
c := −1/10; d := 1;
For[k1 = −d, k1 <= d, k1 + +,
f1[u1_] := N[Sqrt[u1]];
f2[u2_] := N[u2];
xr1 = N[Nest[f1[h2g1[#, #]]&, N[c], 4], 100];
yr2 = N[Nest[f2[h2g2[#, #]]&, N[c], 4], 100];
FQ1 = N[xr1, 15];
FQ2 = N[yr2, 15];
s1 = N[x1, 100]/.FindRoot [{Exp[x1] + 2*y1^2 − 3 == 0, x1^2 + y1*x1 + y1 − 1 == 0} , {x1, FQ1}, {y1, FQ2},
WorkingPrecision− > 450];
s2 = N[y1, 100]/.FindRoot [{Exp[x1] + 2*y1^2 − 3 == 0, x1^2 + y1*x1 + y1 − 1 == 0} , {x1, FQ1}, {y1, FQ2},
WorkingPrecision− > 450];
Print[”(k1=”,k1,”)”,”,”x=”,N[t1,50],”,”,”y=”,N[t2, 50]];
Print[N[g1[t1, t2], 45],”,”,N[g2[t1, t2], 45]]]
```

Result 6. For 1 root with of $\{x, y\}$

(k1 = 0), x = 0.000000000, y = 1.0000000000000

For all cases the approximations are of the order of 10^ − 20 and above.

IV. 2$^{\text{nd}}$. Example(Composite System 2X2): We will solve a 2 × 2 system that is complex and catches all the cases we have mentioned as procedures and methods. We will have a generalized form, difficult, but practicable with its very powerful method.

We assume that we have the system,

$$\left\langle\begin{aligned}&y \cdot \cos(x) + x \cdot \sin(y) = 5\\&e^{x \cdot y} + y \cdot \log(x) + x = 6\end{aligned}\right.$$

IV.1. We use Method 1. In the analysis we have 9 basic categories and 20 subcategories. We apply the trigonometric categorization leading to a radical manifold such also as the following forms of exponential manifold and logarithmic . Finally and with method 3 of exchange the initial values of variables, we have a total of 29 cases that to consider.

1^{st} Category of Roots

$$1.\left\langle\begin{array}{l}\mathrm{Cos}(x)=u\Rightarrow x=-\,\mathrm{ArcCos}[u]+2k\pi\\ y=v\end{array}\right.$$

$$2.\left\langle\begin{array}{l}\mathrm{Cos}(x)=u\Rightarrow x=\mathrm{ArcCos}[u]+2k\pi\\ y=v\end{array}\right.$$

We have 2 forms according to the basic categorization reported and we will have the cases:

IV.1.1. Correct the initial values of variables.

According to the correct determination of initial values mentioned above, we choose as initial value $c = -0.00001$ which is in the interval $(-1, 1)$. We distinguish the general cases:

Case 1. We have 1 form depending on the basic categorization reported that like we have the form,

$$1.\left\langle\begin{array}{l}\mathrm{Cos}(x)=u\Rightarrow x=-\,\mathrm{ArcCos}[u]+2k\pi\\ y=v\end{array}\right.$$

That like we have the program:

Program in mathematica

```
Off[ General::munfl.FindRoot::cvmit];
h2g1[x_, y_] := N [(5 − x* Sin[y]) /y];
h2g2[x_, y_] := −N [(Exp [x*y] + x − 6) / Log[x]];
g1[x_, y_] := N[y* Cos[x] + x* Sin[y] − 5];
g2[x_, y_] := N [Exp[x*y] + y* log[x] + x − 6];
c := {−0.00001}; d := 10;
For[k1 = −d, k1 <= d, k1 + +,
f1[u1_] : = N[− ArcCos[u1] + 2πk1];
f2[u2_] : = N[u2];
If[k1 < 0,Goto [mama1],Goto[mama2]];
Label[mama1];
xr1=N[Nest[f1[h2g1[#, #]]&, N[c], 3], 100];
yr 2 =N[Nest[f2[h2g2[#, #]]&, N[c], 3], 100];
Goto[mama3];
Label[mama2];
xr1=N[Nest[f1[h2g1[#, #]]&, N[c], 3], 100];
yr2=N[Nest[f2[h2g2[#,#]]&,N[c],3],100];
Label [mama3];
FQ1 = N[xr1, 475];
FQ2=N[yr2,475];
s10 = FQ2;
s20=FQ1;
s1 = N[x1, 100]/.FindRoot [{y1* Cos[x1] + x1* Sin[y1] − 5 == 0, Exp[x1*y1] + y1 ∗ log[x1] + x1 − 6 == 0} ,
{x1, s10}, {y1, s20}, Method− >”Newton”,WorkingPrecision− > 550];
s2 = N[y1, 100]/.FindRoot[{y1* Cos[x1] + x1* Sin[y1] − 5 == 0, Exp[x1*y1] + y1 ∗ log[x1] + x1 − 6 == 0},
{x1, s10}, {y1, s20},Method− >”Newton”, WorkingPrecision-> 550];
Print [”(k1”, k1, ”) = ”, ”x = ”, N[s1, 15], ”, y = ”, N[s2, 15], N[g1[s1, s2], 45], ”, ”, N[g2[s1, s2], 45]]]
```

Result 1. For the first 11 roots with Infinite Roots of $\{x, y\}$

```
(k1 = −5) x = {0.49339359020619 − 3.46255106216520 I}, y = {−31.5857633327390 + 5.6889627148943 I}
(k1 = −4) x = {0.55236984187412 − 3.32186796761701 I}, y = {−25.3052775641835 + 5.3744406769994 I}
(k1 = −3) x = {0.72113972902306 − 3.37828842145245 I}, y = {−19.0887775925665 + 5.1382860716794 I}
(k1 = −2) x = {1.02769321360653 − 3.43576380064375 I}, y = {−12.93274521823935 + 4.80077816366230 I}
(k1 = −1) x = {1.31553540733466 + 2.91531829760141 I}, y = {−6.59888939808560 − 3.83288459257139 I}
(k1 = 0) x = {3.90328951754611 + 3.34735829343413 I}, y = {−2.07856890149400 − 2.87126326398533 I}
(k1 = 1) x = {1.27491375800521 − 2.02465392399351 I}, y = {6.71533978625608 − 3.01840681167157 I}
(k1 = 2) x = {0.360631263062360 − 0.3976518663233 I}, y = {11.60187290852736 − 3.31572084751079 I}
(k1 = 3) x = {0.72832096793842 − 2.37479058353811 I}, y = {19.0684401728990 − 4.3975776018967 I}
(k1 = 4) x = {0.68698082310458 − 2.76040689500581 I}, y = {25.3912026749717 − 4.9471269419179 I}
(k1 = 5) x = {0.55060055058244 − 2.63361313077523 I}, y = {31.6047117564224 − 5.0914462113089 I}
```

Infinite Roots

IV.1.2. Case 2. We also have 2nd form depending on the basic categorization reported, and we will have the form,

$$2.\left\langle\begin{array}{l}\mathrm{Cos}(x) = u \Rightarrow x = \mathrm{ArcCos}[u] + 2k\pi\\ y = v\end{array}\right.$$

That like we have the program:

Program in mathematica

```
Off[General::munfl, FindRoot::cvmit];
h2g1[x_, y_] := N[(5 − x* Sin[y])/y]
h2g2[x_, y_] := −N[(Exp[x*y] + x − 6)/ log[x]];
g1[x_, y_] = N [y* Cos[x] + x* Sin[y] − 5];
g2[x_, y_] := N [Exp [x*y] + y* log[x] + x − 6];
c := {−1/10^9} ; d1 := 10;
For[k1 = −d1, k1 <= d1, k1 + +,
f1[u1_] = N[ArcCos[u1] + 2πk1];
f2[u2_] := N[u2];
If[k1 < 0,Goto[mama1],Goto[mama2]];
Label[mama1];
xr1 =N[Nest[f1[h2g1[#, #]]&, N[c], 1], 300];
yr2=N[Nest[f2[h2g2[#, #]]&, N[c], 1], 300];
Goto[mama3];
Label[mama2];
xr1=N[Nest[f1[h2g1[#, #]]&, N[c], 1], 300];
yr2 = N[Nest[f2[h2g2[#, #]]&, N[c], 1], 300];
Label[mama3];
FQ1=N[xr1, 375];
FQ2 = N[yr2, 375]
s11 = N[x1, 25]/.FindRoot[{y1* Cos[x1] + x1* Sin[y1] − 5 == 0, Exp[x1*y1] + y1* Log[x1] + x1 − 6 == 0},
{x1, FQ1}, {y1, FQ2},Method− >”Newton”.WorkingPrecision− > 750];
s21 = N[y1, 25]/.FindRoot[{y1* Cos[x1] + x1* Sin[y1] − 5 == 0, Exp[x1*y1] + y1* Log[x1] + x1 − 6 == 0},
{x1, FQ1}, {y1, FQ2},Method− >”Newton”.WorkingPrecision − > 750];
s1 = N[x1, 25]/.FindRoot[{y1* Cos[x1] + x1* Sin[y1] − 5 == 0, Exp[x1*y1] + y1* Log[x1] + x1 − 6 = 0},
{x1, s11}, {y1, s21},WorkingPrecision− > 750];
s2 = N[y1, 25]/.FindRoot[{y1* Cos[x1] + x1* Sin[y1] − 5 == 0, Exp[x1*y1] + y1* Log[x1] + x1 − 6 == 0},
{x1, s11}, {y1, s21},WorkingPrecision− > 750];
If[Abs[g1[s1, s2]] < 0.00000001&& Abs[Re[g2[s1, s2]]] < 0.00000001,Goto[mam1],Goto[mam2]];Label[mam1];
Print[”(k1=”,k1,”),”,”x=”,N[s1,14],”,y=”,N[s2,14],N[Re[g1[s1,s2]],45],”,”,N[Re[g2[s1,s2]],45]];Label[mam2];]
```

Result 2. For the first 11 roots with Infinite Roots of $\{\mathbf{x},\mathbf{y}\}$

$(\mathrm{k1}=-5),\ \mathrm{x}=\{-30.567939656116-3.999357433163\ \mathrm{I}\},\ \mathrm{y}=\{-0.14379722051789-0.19038669522717\ \mathrm{I}\}$

$(\mathrm{k1}=-4),\ \mathrm{x}=\{-24.261186090320-3.787242577785\ \mathrm{I}\},\ \mathrm{y}=\{-0.17945340358251-0.23617655751124\ \mathrm{I}\}$

$(\mathrm{k1}=-3),\ \mathrm{x}=\{-17.946628537527-3.508198474963\ \mathrm{I}\},\ \mathrm{y}=\{-0.24194077343454-0.31081384579479\ \mathrm{I}\}$

$(\mathrm{k1}=-2),\ \mathrm{x}=\{11.2906344766744-6.24538477976\ \mathrm{I}\},\ \mathrm{y}=\{-2.8506781864347+5.5110065451972\ \mathrm{I}\}$

$(\mathrm{k1}=-1),\ \mathrm{x}=\{-5.2239900434204-5.62254740692\ \mathrm{I}\},\ \mathrm{y}=\{-6.9250350514790-5.7835948207647\ \mathrm{I}\}$

$(\mathrm{k1}=0),\ \mathrm{x}=\{4.7219883866462-4.1331757921678\ \mathrm{I}\},\ \mathrm{y}=\{-2.9784092669427+3.8539793137187\ \mathrm{I}\}$

$(\mathrm{k1}=1),\ \mathrm{x}=\{14.965286976222\},\ \mathrm{y}=\{-3.3134407289362\}$

$(\mathrm{k1}=2),\ \mathrm{x}=\{18.557261277031-1.919909073080\ \mathrm{I}\},\ \mathrm{y}=\{-4.3091097774017+0.5043000104099\ \mathrm{I}\}$

$(\mathrm{k1}=3),\ \mathrm{x}=\left\{24.766482373690+0.{}^{*}10^{-13}\ \mathrm{I}\right\},\ \mathrm{y}=\{-5.8471829571873+0.{}^{*}10-14\ \mathrm{I}\}$

$(\mathrm{k1}=4),\ \mathrm{x}=\{27.540210478612+1.488112596898\ \mathrm{I}\},\ \mathrm{y}=\{-6.4992573579945-0.3428507117888\ \mathrm{I}\}$

$(\mathrm{k1}=5),\ \mathrm{x}=\{27.540210478612+1.488112596898\ \mathrm{I}\},\ \mathrm{y}=\{-6.4992573579945-0.3428507117888\ \mathrm{I}\}$

Infinite Roots

We notice in both cases that we are also given real roots. Of course we also have complex roots. We will extract the total plural roots in the last category that we will examine.

IV.1.3. Exchange the initial values of variables. (We use Method 3)

For these 2 cases i continue with Method 3 ie, i make exchange the initial values of variables, i.e., where $x_{in} = y_0 \wedge y_{in} = x_0$ and thus calculate again the values of the variables for the system. Therefore, i can make the variables symmetrical and to i find other roots that are likely to exist. The values of treatment $x_{in} = y_0 \wedge y_{in} = x_0$ are meant with initial values of the first approximation by **of the method GRIM** of (Nest - function)

IV.1.3.1. Case 1. We have 1 form depending on the basic categorization reported that like we have the form with **Exchange the initial values of variables**

$$3.\ \begin{cases}\mathrm{Cos}(x) = u \Rightarrow x = -\,\mathrm{ArcCos}[u] + 2k\pi \\ y = v \\ x_{in} = y_0 \wedge y_{in} = x_0\end{cases}$$

That like we have the Program:

Program in mathematica

```
Off[General::munfl. FindRoot::cvmit];
h2g1[x_, y_] == N[(5 − x* Sin[y])/y];
h2g2[x_, y_] := −N[(Exp[x*y] + x − 6)/ Log[x]];
g1[x_, y_]] = N [y* Cos[x] + x* Sin[y] − 5]
g2[x_, y_] := N [Exp [x*y] + y* Log[x] + x − 6];
c := {−0.00001}; d1 := 10
For[k1 = −d1, k1 <= d1, k1 + +,
f1[u1_] := N[−ArcCos[u1] + 2πk1];
f2[u2_] := N[u2];
If[k1<0,Goto[mama1],Goto[mama2]];
Label[mama1];
xr1 = N[Nest[f1[h2g1[#, #]]&, N[c], 1], 300];
yr2 = N[Nest[f2[h2g2[#, #]]&, N[c], 1], 300];
Goto[mama3];
Label[mama2];
xr1=N[Nest[f1[[h2g1[#,#]]&,N[c],1],300];
yr2=N[Nest[f2[h2g2[#,#]]&,N[c],1],300];
Label[mama3];
FQ1 = N[xr11, 375]
FQ2 = N[yr2, 375]
s11 = N[x1, 25]/.FindRoot [{y1* Cos[x1] + x1* Sin[y1] − 5 == 0, Exp[x1*y1] + y1* Log[x1] + x1 − 6 == 0},
{x1, FQ2}, {y1, FQ1}, Method− >”Newton”.WorkingPrecision− > 750];
s21 = N[y1, 25]/.FindRoot[{y1* Cos[x1] + x1* Sin[y1] − 5 == 0, Exp[x1*y1] + y1* Log[x1] + x1 − 6 == 0},
{x1, FQ2}, {y1, FQ1},Method− >”Newton”.WorkingPrecision− > 750];
s1 = N[x1, 25]/.FindRoot[{y1* Cos[x1] + x1* Sin[y1] − 5 == 0, Exp[x1*y1] + y1* Log[x1] + x1 − 6 == 0},
{x1, s11}, {y1, s21}, WorkingPrecision− > 750];
s2 = N[y1, 25]/.FindRoot[{y1* Cos[x1] + x1* Sin[y1] − 5 == 0, Exp[x1*y1] + y1* log[x1] + x1 − 6 == 0},
{x1, s11}, {y1, s21}, WorkingPrecision− > 750];
If[Abs[g1[s1, s2]] < 0.00000001 && Abs[g1[s1, s2]] > 0,Goto[mam1],Goto[mam2]];
Label[mam1]; Print[”(k1=”,k1,”)”,”,”x =”,N[s1, 14], ”, y =”, N[s2, 14]];Label[mam2];]
```

Result 3. For the first 11 roots with Infinite Roots of $\{x, y\}$

```
(k1 = −5), x = {−0.15672551765505 + 0.10241125007137 I}, y = {−33.424219466919 + 6.023675748785 I}
(k1 = −4), x = {−0.18763149188740 + 0.12531762602468 I}, y = {−27.138394520659 + 5.659764334811 I}
(k1 = −3), x = {−0.23646582762954 + 0.16164077935723 I}, y = {−20.853169789957 + 5.205258470892 I}
(k1 = −2), x = {−0.32698644765776 + 0.22780617192964 I}, y = {−14.573484728880 + 4.598429674131 I}
(k1 = −1), x = {0.64023848861854 − 0.102433241853520 I}, y = {−10.9869997223966 + 3.7853751637628 I}
(k1 = 0), x = {−0.81207408344963 − 0.08955624153056 I}, y = {−1.1274634548844 + 2.6712918349786 I}
(k1 = 1), x = {0.12077983506281 − 0.705663099230330 I}, y = {−5.8084971363285 + 3.6085759819732 I}
(k1 = 2), x = {0.70032692005390 − 1.678528196362400 I}, y = {12.6275239179040 − 3.5230390648396 I}
```

```
(k1 = 3),  x = {−1.015418351770500 − 3.8035596671704 I},  y = {16.156077273272 + 5.255537716045 I}
(k1 = 4),  x = {−0.0952119776106 − 1.483199705816900 I},  y = {28.135140398750 + 4.406930226079 I}
(k1 = 5),  x = {−0.01585700273962 − 1.03324253598674 I},  y = {47.010297040625 + 4.906596837592 I}
```

Infinite Roots

IV.1.3.2. Case 2. We have 1 form depending on the basic categorization reported that like we have the form with **Exchange the initial values of variables**.

$$4.\ \begin{cases} \mathrm{Cos}(x) = u \Rightarrow x = \mathrm{Cos}[u] + 2k\pi \\ y = v \\ x_{in} = y_0 \wedge y_{in} = x_0 \end{cases}$$

That like we have the Program:

Program in mathematica

```
Off General::munfl.FindRoot::cvmit];
h2g1[x_, y_] := N[(5 − x* Sin[y])/y]
h2g2[x_, y_] = −N [(Exp [x*y] + x − 6) / log[x]] ;
g1[x_, y_] := N [y* Cos[x] + x* Sin[y] − 5]
g2[x_, y_] := N [Exp [x*y] + y* log[x] + x − 6]
c := {−0.00001}; d1 := 10;
For[k1 = −d1, k1 <= d1, k1 + +,
f1[u1_] := N[ArcCos[u1] + 2πk1];
f2[u2_] := N[u2];
If[k1 < 0,Goto[mama1],Goto[mama2]];
Label[mama1];
xr1=N[Nest[f1[h2g1[#,#]]&,N[c],1],300];
yr2= N[Nest[f2[h2g2[#, #]]&, N[c], 1], 300];
Goto[mama3];
Label[mama2];
xr1=N[Nest[f1[h2g1[#, #]]&, N[c], 1], 300];
yr2= N[Nest[f2[h2g2[#, #]]&, N[c], 1], 300];
Label[mama3];
FQ1=N[xr1,375];
FQ2 = N[yr2, 375];
s11 = N[x1, 25]/.FindRoot[{y1* Cos[x1] + x1* Sin[y1] − 5 == 0, Exp[x1*y1] + y1* Log[x1] + x1 − 6 == 0},
{x1, FQ2}, {y1, FQ1},Method− >”Newton”.WorkingPrecision− > 750];
s21=N[y1,25]/.FindRoot[{y1* Cos[x1] + x1* Sin[y1] − 5 == 0, Exp[x1*y1] + y1* Log[x1] + x1 − 6 == 0} ,
{x1, FQ2}, {y1, FQ1},Method− >”Newton”,WorkingPrecision− > 750];
s1 = N[x1, 25]/.FindRoot[{y1* Cos[x1] + x1* Sin[y1] − 5 == 0, Exp[x1*y1] + y1* Log[x1] + x1 − 6 == 0},
{x1, s11}, {y1, s21},WorkingPrecision− > 750];
s2 = N[y1, 25]./FindRoot[{y1* Cos[x1] + x1* Sin[y1] − 5 == 0, Exp[x1*y1] + y1* Log[x1] + x1 − 6 == 0},
{x1, s11}, {y1, s21},WorkingPrecision− > 750];
If [Abs[g1[s1, s2]] < 0.00000001&& Abs[g1[s1, s2]] > 0,Goto[mam1],Goto[mam2]];
Label[mam1];
Print[”(k1=”,k1,”),”, x = ”,  N[ s1, 14], ”, y = ”, N[s2,14]];
```

```
Print[N[Re[g1[s1,s2]],45],”,”,N[Re[g2[s1,s2]],45]];
Label[mam 2]; ]
```

Result 4. For the first 11 roots with Infinite Roots of $\{x, y\}$

$(\mathrm{k1} = -5),\ \mathrm{x} = \{-0.21538553182512 - 0.35386767145274\ \mathrm{I}\},\ \mathrm{y} = \{-27.634855761886 - 5.107981479097\ \mathrm{I}\}$
$(\mathrm{k1} = -4),\ \mathrm{x} = \{0.80621595367466 + 0.03033523218618\ \mathrm{I}\},\ \mathrm{y} = \{-23.495103336024 - 3.974408951970\ \mathrm{I}\}$
$(\mathrm{k1} = -3),\ \mathrm{x} = \{-0.32698644765776 - 0.22780617192964\ \mathrm{I}\},\ \mathrm{y} = \{-14.573484728880 - 4.598429674131\ \mathrm{I}\}$
$(\mathrm{k1} = -2),\ \mathrm{x} = \{0.64023848861854 + 0.10243324185352\ \mathrm{I}\},\ \mathrm{y} = \{-10.9869997223966 - 3.78537516376\ \mathrm{I}\}$
$(\mathrm{k1} = -1),\ \mathrm{x} = \{2.02188851914830 + 3.83329370917930\ \mathrm{I}\},\ \mathrm{y} = \{-7.2372191454003 - 4.512986395104\ \mathrm{I}\}$
$(\mathrm{k1} = 0),\ \mathrm{x} = \{9.871234891505100 + 3.0287572691571\ \mathrm{I}\},\ \mathrm{y} = \{-1.7944475001413 - 1.068502123119\ \mathrm{I}\}$
$(\mathrm{k1} = 1),\ \mathrm{x} = \{4.984120565956200 - 3.7790506179944\ \mathrm{I}\},\ \mathrm{y} = \{3.1858653276498 - 3.531769543940\ \mathrm{I}\}$
$(\mathrm{k1} = 2),\ \mathrm{x} = \{1.527735372224100 - 3.5401086138841\ \mathrm{I}\},\ \mathrm{y} = \{13.358277700131 - 4.835800446585\ \mathrm{I}\}$
$(\mathrm{k1} = 3),\ \mathrm{x} = \{0.484922161355100 - 1.4902215418046\ \mathrm{I}\},\ \mathrm{y} = \{18.804642838136 - 3.908899028725\ \mathrm{I}\}$
$(\mathrm{k1} = 4),\ \mathrm{x} = \{0.37464691559130 - 1.38158463530990\ \mathrm{I}\},\ \mathrm{y} = \{25.052548421478 - 4.206383563218\ \mathrm{I}\}$
$(\mathrm{k1} = 5),\ \mathrm{x} = \{0.30742602165908 - 1.31200415852701\ \mathrm{I}\},\ \mathrm{y} = \{31.323877886673 - 4.443589889106\ \mathrm{I}\}$

Infinite Roots

For all cases the approximations are of the order of $10^\wedge - 20$ and above.
With the technique of exchange initial values we see that for the same category we find a number of new roots. Exactly the same happens in every case where we create an exchange of initial values. Here applies the 2! = 2 cases type of event we did.

V.2. 2^{nd} Category of Roots

We have 2 forms according to the basic categorization mentioned and we will have 4 cases with exchange. In this category we use **Newton's form of approximation** by groups in **5-Multiple filters**. This is because the project flow in fewer groups approach does not correspond well to the programmatic approach.

1. $\left\langle \begin{array}{l} \mathrm{Sin}(y) = u \Rightarrow y = \mathrm{ArcSin}[u] + 2k\pi \\ x = v \end{array} \right.$

2. $\left\langle \begin{array}{l} \mathrm{Sin}(y) = u \Rightarrow y = \pi - \mathrm{ArcSin}[u] + 2k\pi \\ x = v \end{array} \right.$

V.2.1. Case 1. We have 1 form depending on the basic categorization, therefore we have the form,

1. $\left\langle \begin{array}{l} \mathrm{Sin}(y) = u \Rightarrow y = \mathrm{ArcSin}[u] + 2k\pi \\ x = v \end{array} \right.$

That like we have the program:

Program in mathematica

```
Off[General::munfl.FindRoot::cvmit];
h2g2[x_, y_] := N[(5 − y* Cos[x])/x]
h2g1[x_, y_] := −N [(Exp [x*y] + y* Log[x] − 6)]
```

```
g1 [x_y_] := N [y* Cos[x] + x* Sin[y] − 5]
g2[x_, y_] := N [Exp [x*y] + y* Log[x] + x − 6]
c := {−3}; d := 10
c1:= {−3}
For[k1 = −d, k1 <= d, k1 + +,
f1[u1_] := N[ArcSin[u1] + 2πk1];
f2[u2_] := N[u2];
If[k1 < 0,Goto[mama1],Goto[mama2]];
Label[mama1];
xr1=N[Nest[f1[h2g1[#, #]]&, N[c1], 1], 300];
yr2=N[Nest[f2[h2g2[#,#]]&,N[c1],1],300];
Goto[mama3];
Label[mama2];
xr1 = N[Nest[f1[h2g1[#, #]]&, N[c], 1], 300];
yr2= N[Nest[f2[h2g2[#, #]]&, N[c], 1], 300];
Label[mama3];
FQ1 = N[xr11, 475]
FQ2 = N[yr2, 475];
s10 = N[x1, 100]/.FindRoot[{y1* Cos[x1] + x1* Sin[y1] − 5 == 0, Exp[x1*y1] + y1* Log[x1] + x1 − 6 == 0},
{x1, FQ1}, {y1, FQ2},WorkingPrecision− > 750,WorkingPrecision− > 250];
s20 = N[y1, 100]/.FindRoot [{y1* Cos[x1] + x1* Sin[y1] − 5 == 0, Exp[x1*y1] + y1* Log[x1] + x1 − 6 == 0} ,
{x1, FQ1}, {y1, FQ2},WorkingPrecision− > 750,WorkingPrecision− > 250];
s101 = N[x1, 100]/.FindRoot[{y1* Cos[x1] + x1* Sin[y1] − 5 == 0, Exp[x1*y1] + y1* Log[x1] + x1 − 6 == 0},
{x1,  s10}, {y1, s20},WorkingPrecision− > 750,WorkingPrecision− > 250];
s201= N[y1, 100]/.FindRoot[{y1* Cos[x1] + x1* Sin[y1] − 5 == 0, Exp[x1*y1] + y1* Log[x1] + x1 − 6 == 0},
{x1,  s10}, {y1, s20},WorkingPrecision− > 750,WorkingPrecision− > 250];
s102 = N[x1, 100]/.FindRoot[{y1* Cos[x1] + x1* Sin[y1] − 5 == 0, Exp[x1*y1] + y1* Log[x1] + x1 − 6 == 0} ,
{x1, s101}, {y1, s201},WorkingPrecision− > 750,WorkingPrecision− > 250];
s202= N[y1, 100]/.FindRoot[{y1* Cos[x1] + x1* Sin[y1] − 5 == 0, Exp[x1*y1] + y1* Log[x1] + x1−
6 == 0}, {x1, s101}, {y1, s201},WorkingPrecision− > 750,WorkingPrecision− > 250];
s103 = N[x1, 100]/.FindRoot[{y1* Cos[x1] + x1* Sin[y1] − 5 == 0, Exp[x1*y1] + y1* Log[x1] + x1−
6 == 0}, {x1, s102}, {y1, s202},WorkingPrecision− > 750,WorkingPrecision− > 250];
s203=N[y1,100]/.FindRoot[{y1* Cos[x1] + x1* Sin[y1] − 5 == 0, Exp[x1*y1] + y1* Log[x1] + x1 − 6 == 0},
{x1, s102}, {y1, s202},WorkingPrecision− > 750,WorkingPrecision− > 250];
s1 = N[x1, 100]/.FindRoot[{y1* Cos[x1] + x1* Sin[y1] − 5 == 0, Exp[x1 ∗ y1] + y1* Log[x1] + x1 − 6 == 0} ,
s203 = N[y1, 100]/.FindRoot[{y1* Cos[x1] + x1* Sin[y1] − 5 == 0, Exp[x1*y1] + y1* Log[x1] + x1 − 6 == 0},
{x1, s102}, {y1, s202}, WorkingPrecision− > 750, WorkingPrecision− > 250];
s1 = N[x1, 100]/.FindRoot[{y1* Cos[x1] + x1* Sin[y1] − 5 == 0, Exp[x1*y1] + y1* Log[x1] + x1 − 6 == 0},
{x1, s103}, {y1, s203},WorkingPrecision− > 750,WorkingPrecision− > 250];
s2 = N[y1, 100]/.FindRoot[{y1* Cos[x1] + x1* Sin[y1] − 5 == 0, Exp[x1*y1] + y1∗ Log[x1] + x1 − 6 == 0},
{x1, s103}, {y1, s203},WorkingPrecision− > 750,WorkingPrecision− > 250];
Print[”(k1 = ”, k1, ”) ,” , x” = ”, N[x1, 14], ”, y = ”, N[ s2, 14]]
Print[N[g1[s1,s2],45],”,”,N[g2[s1,s2],45]]]
```

Result 5. For the first 11 roots with Infinite Roots of $\{x, y\}$

(k1 = −5), x = {−30.567939656116 + 3.999357433163 I}, y = {−0.14379722051789 + 0.19038669522717 I}
(k1 = −4), x = {−40.832043798563 + 3.195732428237 I}, y = {−0.09396441190092 − 0.00585732392240 I}
(k1 = −3), x = {−29.110000655948 − 7.724067365654 I}, y = {4.888493179063901 + 6.40621956948950 I}
(k1 = −2), x = {−19.404496172607 − 3.601291605717 I}, y = {−0.10275336166801 + 0.33811052237396 I}
(k1 = −1), x = {−11.6202455625454 + 3.0958439219 I}, y = {−0.37902656913383 + 0.45318071992616 I}
(k1 = 0), x = {−1.7906090735446 + 0.958637370083 I}, y = {−2.5810420664808 + 1.9628613906574 I}
(k1 = 1), x = {9.8712348915051 + 3.0287572691571 I}, y = {−1.7944475001413 − 1.0685021231195 I}
(k1 = 2), x = {14.965286976222}, y = {−3.3134407289362}
(k1 = 3), x = {18.557261277031 + 1.919909073080 I}, y = {−4.3091097774017 − 0.5043000104099 I}
(k1 = 4), x = {24.766482373690} ,y = {−5.8471829571873}
(k1 = 5), x = {27.540210478612 − 1.488112596898 I}, y = {−6.4992573579945 + 0.3428507117888 I}
Infinite Roots

V.2.2. Case 2. We have 1 form depending on the basic categorization, therefore we have the form,

$$2.\ \left\langle\begin{array}{l}\mathrm{Sin}(y)=u\Rightarrow y=\pi-\mathrm{ArcSin}[u]+2k\pi\\ x=v\end{array}\right.$$

that like we have the program:
Program in mathematica

```
Off[General::munfl,FindRoot::cvmit]:
h2g2[x_, y_] := N [(5 − y* Cos[x]) /x];
h2g1[x_, y_] := −N [(Exp [x*y] + y* Log[x] − 6)];
g1[x, y] := N[y* Cos[x] + x* Sin[y] − 5];
g2[x, y] := N [Exp [x*y] + y* Log[x] + x − 6]
c := {−3}; d := 10;
cl:= {−3}
For[k1 = −d, k1 <= d, k1 + +,
f1[u1_] = N[π − ArcSin[u1] + 2πk1];
f2[u2_] := N[u2];
If[k1 < 0,Goto[mama1],Goto[mama2]];
Label[mama1];
xr1=N[Nest[f1[h2g1[#,#]]&,N[c1],1],300];
yr 2 = N[Nest[f2[h2g2[#, #]]&, N[c1], 1], 300];
Goto[mama3];
Label[mama2];
xr1 = N[Nest[f1[h2 g1[#, #]]&, N[c], 1], 300];
yr2=N[Nest[f2[h2g2[#, #]]&, N[c], 1], 300];
Label[mama3];
FQ1 = N[xr1, 475];
FQ2 = N[yr2, 475];
s10 = N[x1, 100]/.FindRoot[{y1* Cos[x1] + x1* Sin[y1] − 5 == 0, Exp[x1*y1] + y1* Log[x1] + x1 − 6 == 0},
{x1, FQ1}, {y1, FQ2},WorkingPrecision− > 750,WorkingPrecision− > 250];
s20 = N[y1, 100]/.FindRoot[{y1 * Cos[x1] + x1* Sin[y1] − 5 == 0, Exp[x1*y1] + y1* Log[x1] + x1 − 6 == 0},
{x1, FQ1}, {y1, FQ2},WorkingPrecision− > 750,WorkingPrecision− > 250];
s101 = N[x1, 100]/.FindRoot[{y1* Cos[x1] + x1* Sin[y1] − 5 == 0, Exp[x1*y1] + y1* Log[x1] + x1 − 6 == 0},
{x1, s10}, {y1, s20},WorkingPrecision− > 750,WorkingPrecision− > 250];
s201=N[y1,100]/.FindRoot[{y1* Cos[x1] + x1* Sin[y1] − 5 == 0, Exp[x1*y1] + y1* Log[x1] + x1 − 6 == 0} ,
{x1, s10}, {y1, s20},WorkingPrecision− > 750,WorkingPrecision− > 250];
s102 = N[x1, 100]/ FindRoot [{y1* Cos[x1] + x1* Sin[y1] − 5 == 0, Exp[x1*y1] + y1* Log[x1] + x1−
6 == 0}, {x1, s101}, {y1, s201},WorkingPrecision− > 750,WorkingPrecision− > 250];
s202 = N[y1, 100]/.FindRoot[{y1* Cos[x1] + x1* Sin[y1] − 5 == 0, Exp[x1*y1] + y1* Log[x1] + x1 − 6 == 0},
{x1, s101}, {y1, s201},WorkingPrecision− > 750,WorkingPrecision− > 250];
s103 = N[x1, 100]/.FindRoot[{y1* Cos[x1] + x1* Sin[y1] − 5 == 0, Exp[x1*y1] + y1* Log[x1] + x1 − 6 == 0},
{x1, s102}, {y1, s202},WorkingPrecision− > 750,WorkingPrecision− > 250];
s203 = N[y1, 100]/.FindRoot[{y1* Cos[x1] + x1* Sin[y1] − 5 == 0, Exp[x1*y1] + y1* Log[x1] + x1 − 6 == 0},
{x1, s102}, {y1, s202},WorkingPrecision− > 750,WorkingPrecision− > 250];
s1 = N[x1, 100]/.FindRoot[{y1* Cos[x1] + x1* Sin[y1] − 5 == 0, Exp[x1*y1] + y1* Log[x1] + x1 − 6 == 0},
{x1, s103}, {y1, s203},WorkingPrecision− > 750,WorkingPrecision− > 250];
s2 = N[y1, 100]/.FindRoot[{y1* Cos[x1] + x1* Sin[y1] − 5 == 0, Exp[x1*y1] + y1* Log[x1] + x1 − 6 == 0},
{x1, s103}, {y1, s203},WorkingPrecision− > 750,WorkingPrecision− > 250];
Print["(k1=",k1,")",","x=",N[s1,15],",y=",N[s2,15]];
Print[N[g1[s1, s2], 45], ", ", N[g2[s1, s2], 45]]]
```

Result 6. For the first 11 roots with any value of $\{x, y\}$

(k1 = −5), x = {−28.2402092211935 + 3.1526908003804 I}, y = {−0.1243853190860 − 0.0110910650448439 I}
(k1 = −4), x = {−29.1100006559480 + 7.7240673656542 I}, y = {4.88849317906390 − 6.4062195694895010 I}
(k1 = −3), x = {−19.4044961726070 + 3.6012916057173 I}, y = {−0.102753361668007 − 0.33811052237396 I}
(k1 = −2), x = {−11.62024556254537 − 3.0958439219303 I}, y = {−0.37902656913383 − 0.453180719926156 I}
(k1 = −1), x = {−1.79060907354458 − 0.95863737008391 I}, y = {−2.58104206648083 − 1.96286139065743 I}
(k1 = 0), x = {18.55726127703140 − 1.91990907307990 I},y = {−4.30910977740174 + 0.50430001040989 I}
(k1 = 1), x = {14.9652869762224} ,y = {−3.31344072893624}
(k1 = 2), x = {18.55726127703140 − 1.91990907307990 I}, y = {−4.30910977740174 + 0.50430001040989 I}
(k1 = 3), x = {24.7664823736900}, y = {−5.84718295718732}
(k1 = 4), x = {27.5402104786121 + 1.48811259689820 I}, y = {−6.49925735799447 − 0.34285071178880 I}
(k1 = 5), x = {34.6611764554317 + 2.36607103929210 I}, y = {−8.08807401969845 − 0.51151055350744 I}
Infinite Roots

V.3. 3^{rd} Category of Roots

We have 2 forms according to the basic categorization mentioned and we will have 4 cases with exchange. In this category we use **Newton's form of approximation** by groups in **1-Multiple filters**.

1. $$\left\langle \begin{array}{l} \mathrm{Sin}(y) = u \Rightarrow y = \mathrm{ArcSin}[u] + 2k\pi \\ \mathrm{Log}(x) = v \Rightarrow x = e^{v} \end{array} \right.$$

2. $$\left\langle \begin{array}{l} \mathrm{Sin}(y) = u \Rightarrow y = \pi - \mathrm{ArcSin}[u] + 2k\pi \\ \mathrm{Log}(x) = v \Rightarrow x = e^{v} \end{array} \right.$$

That like we have the program:

Program in mathematica:

V.3.1 Case 1. We have 1 form depending on the basic categorization, therefore we have the form,

$$1.\ \left\langle \begin{array}{l} \mathrm{Sin}(y) = u \Rightarrow y = \mathrm{ArcSin}[u] + 2k\pi \\ \mathrm{Log}(x) = v \Rightarrow x = e^{v} \end{array} \right.$$

That like we have the program:

Program in mathematica

```
Off[General::munfl.FindRoot::cvmit];
h2g2[x_, y_] := N[(5 − y* Cos[x])/x]
h2g1[x_, y_] := −N [(Exp [x*y] + x − 6) /y]
g1[x_, y_] := N[y* Cos[x] + x* Sin[y] − 5]
g2[x_, y_] := N[Exp[x*y] + y* Log[x] + x − 6];
c := {−2}; d := 10;
For[k1 = −d, k1 <= d, k1 + +,
f1[u1_] := N[ArcSin[u1] + 2πk1];
f2[u2_] := N[Exp[u2]];
If[k1 < 0,Goto[mama1],Goto[mama2]];
Label[mama1];
xr1=N[Nest[f1[h2g1[#, #]]&, N[c], 1], 300];
yr2=N[Nest[f2[h2g2[#,#]]&,N[c],1],300];
Goto[mama3];Label[mama2];
xr1=N[Nest[f1[h2g1[#,#]]&,N[c],1],300];
yr2=N[Nest[f2[h2g2[#,#]]&,N[c],1],300];
Label[mama3];
FQ1 = N[xr1, 375];
FQ2 = N[yr2, 375];
s1 = N[x1, 100]/.FindRoot[{y1* Cos[x1] + x1* Sin[y1] − 5 == 0, Exp[x1*y1] + y1* Log[x1] + x1 − 6 == 0},
{x1, FQ1}, {y1, FQ2},WorkingPrecision− > 650];
s2 =N[y1,100]/.FindRoot[{y1* Cos[x1] + x1* Sin[y1] − 5 = 0, Exp[x1*y1] + y1* Log[x1] + x1 − 6 = 0},
{x1, FQ1}, {y1, FQ2},WorkingPrecision− > 650];
Print[”(k1=”,k1,”),”,”x=”,N[s 1,14],”,y=”,N[s 2,14]]]
```

Result 7. For the first 11 roots with Infinite Roots of $\{x, y\}$

```
(k1 = −5), x = {−28.240209221193 − 3.152690800380 I}, y = {−0.124385319086068 + 0.0110910650448 I}
(k1 = −4), x = {−17.946628537527 − 3.508198474963 I}, y = {−0.24194077343454 − 0.31081384579479 I}
(k1 = −3), x = {−15.610260481764 − 2.978644522260 I}, y = {−0.19279205488658 + 0.03005548052570 I}
(k1 = −2), x = {−9.2319815063655 − 2.763894181135 I}, y = {−0.27757357561001 + 0.07061808552228 I}
(k1 = −1), x = {−2.5065032563201 − 2.239559090232 I}, y = {−0.47990217394887 + 0.39950301840945 I}
(k1 = 0), x = {0.41396940961811 + 0.*10^−15 I}, y = {5.7074792232379 + 0.*10^−14 I}
```

```
(k1 = 1),  x = {4.5501190011626 − 3.3284815835014 I},  y = {0.03914909282626 + 0.28640652026055 I}
(k1 = 2),  x = {4.5501190011626 + 3.328481583501 I},  y = {0.03914909282626 − 0.28640652026055 I}
(k1 = 3),  x = {4.5501190011626 + 3.328481583501 I},  y = {0.03914909282626 − 0.28640652026055 I}
(k1 = 4),  x = {26.865380732088 − 3.8247260577970 I},  y = {0.12957622631877 − 0.10573162077698 I}
(k1 = 5),  x = {35.775858180188 + 3.984441794612 I},  y = {0.084902258654012 − 0.0938880877314 I}
```

Infinite Roots

As we can see there are some repetitions but that does not bother us it is a matter of a specific position. Also we have few real roots.

V.3.2 Case 2. Also we have 1 form depending on the basic categorization, therefore we have the form,

$$2. \left\langle \begin{array}{l} \mathrm{Sin}(y) = u \Rightarrow y = \pi - \mathrm{ArcSin}[u] + 2k\pi \\ \mathrm{Log}(x) = v \Rightarrow x = e^{v} \end{array} \right.$$

That like we have the program:

Program in mathematica

```
Off[General::munfl,FindRoot::cymit];
h2g2[x_, y_] := N[(5 − y* Cos[x])/x]
h2g1[x_, y_] := −N [(Exp [x*y] + x − 6) /y]
g1[x, y] := N [y* Cos[x] + x* Sin[y] − 5]
g2[x_, y_] := N [Exp [x*y] + y* Log[x] + x − 6]
c := {−2}; d := 10;
For[k1 = −d, k1 <= d, k1 + +,
f1[u1_] := N[π − ArcSin[u1] + 2πk1];
f2[u2_] := N[Exp[u2]];
If[k1 < 0,Goto[mama1],Goto[mama2]];
Label[mama1];
xr1=N[Nest[f1[h2g1[#,#]]&,N[c],1],300];
yr2=N[Nest[f2[h2g2[#,#]]&,N[c],1],300];
Goto[mama3];
Label[mama2];
xr1=N[Nest[f1[h2g1[#,#]]&,N[c],1],300];
yr2= N[Nest[f2[h2g2[#, #]]&, N[c], 1], 300]; Label[mama3];
FQ1 = N[xr1, 375];
FQ2=N[yr2,375];
s1 = N[x1, 100]/.FindRoot[{y1* Cos[x1] + x1* Sin[y1] − 5 == 0, Exp[x1*y1] + y1* Log[x1] + x1 − 6 == 0},
{x1, FQ1}, {y1, FQ2},WorkingPrecision− > 650];
s2 = N[y1, 100]/.FindRoot[{y1* Cos[x1] + x1* Sin[y1] − 5 == 0, Exp[x1*y1] + y1* Log[x1] + x1 − 6 == 0},
{x1, FQ1}, {y1, FQ2},WorkingPrecision− > 650];
Print["(k1=",k1,")",",","x=",N[s1,14],",y=",N[s2,14]]]
```

Result 8. For the first 11 roots with Infinite Roots of $\{x, y\}$

$(\mathrm{k1} = -5),\ \mathrm{x} = \{-28.240209221193 + 3.152690800380\ \mathrm{I}\},\ \mathrm{y} = \{-0.124385319086068 - 0.01109106504484\ \mathrm{I}\}$

$(\mathrm{k1} = -4),\ \mathrm{x} = \{-17.946628537527 + 3.508198474963\ \mathrm{I}\},\ \mathrm{y} = \{-0.24194077343454 + 0.31081384579479\ \mathrm{I}\}$

$(\mathrm{k1} = -3),\ \mathrm{x} = \{-15.610260481764 + 2.978644522260\ \mathrm{I}\},\ \mathrm{y} = \{-0.19279205488658 - 0.03005548052570\ \mathrm{I}\}$

$(\mathrm{k1} = -2),\ \mathrm{x} = \{-9.2319815063655 + 2.763894181135\ \mathrm{I}\},\ \mathrm{y} = \{-0.27757357561001 - 0.07061808552228\ \mathrm{I}\}$

$(\mathrm{k1} = -1),\ \mathrm{x} = \{-2.5065032563201 + 2.239559090232\ \mathrm{I}\},\ \mathrm{y} = \{-0.47990217394887 - 0.39950301840945\ \mathrm{I}\}$

$(\mathrm{k1} = 0),\ \mathrm{x} = \left\{0.41396940961811 + 0.{}^{*}10^{-15}\mathrm{I}\right\},\ \mathrm{y} = \left\{5.7074792232379 + 0.{}^{*}10^{-14}\ \mathrm{I}\right\}$

$(\mathrm{k1} = 1),\ \mathrm{x} = \{4.5501190011626 + 3.3284815835014\ \mathrm{I}\},\ \mathrm{y} = \{0.03914909282626 - 0.28640652026055\ \mathrm{I}\}$

$(\mathrm{k1} = 2),\ \mathrm{x} = \{4.5501190011626 - 3.3284815835014\ \mathrm{I}\},\ \mathrm{y} = \{0.03914909282626 + 0.28640652026055\ \mathrm{I}\}$

$(\mathrm{k1} = 3),\ \mathrm{x} = \{4.5501190011626 - 3.3284815835014\ \mathrm{I}\},\ \mathrm{y} = \{0.03914909282626 + 0.28640652026055\ \mathrm{I}\}$

$(\mathrm{k1} = 4),\ \mathrm{x} = \{26.865380732088 + 3.8247260577970\ \mathrm{I}\},\ \mathrm{y} = \{0.12957622631877 + 0.10573162077698\ \mathrm{I}\}$

$(\mathrm{k1} = 5),\ \mathrm{x} = \{35.775858180188 - 3.9844417946120\ \mathrm{I}\},\ \mathrm{y} = \{0.084902258654012 + 0.0938880877314\ \mathrm{I}\}$

Infinite Roots

V.3.3 Case 3. We have 1 form depending on the basic categorization, therefore we have the form with **Exchange the initial values of variables**.

$$3.\ \begin{cases} \mathrm{Sin}(y) = u \Rightarrow y = \mathrm{ArcSin}[u] + 2k\pi \\ \mathrm{Log}(x) = v \Rightarrow x = e^{v} \\ x_{in} = y_0 \wedge y_{in} = x_0 \end{cases}$$

Result 9. For the first 11 roots with Infinite Roots of $\{x, y\}$

$(\mathrm{k1} = -5),\ \mathrm{x} = \{0.84352397325401 + 0.02021128132134\ \mathrm{I}\},\ \mathrm{y} = \{-29.778254735429 - 4.080262802814\ \mathrm{I}\}$

$(\mathrm{k1} = -4),\ \mathrm{x} = \{0.80621595367466 + 0.03033523218618\ \mathrm{I}\},\ \mathrm{y} = \{-23.495103336024 - 3.9744089519701\ \mathrm{I}\}$

$(\mathrm{k1} = -3),\ \mathrm{x} = \{0.74671026871713 + 0.05089579895894\ \mathrm{I}\},\ \mathrm{y} = \{-17.221081346081 - 3.868985427445\ \mathrm{I}\}$

$(\mathrm{k1} = -2),\ \mathrm{x} = \{0.64023848861854 + 0.10243324185352\ \mathrm{I}\},\ \mathrm{y} = \{-10.9869997223966 - 3.78537516376\ \mathrm{I}\}$

$(\mathrm{k1} = -1),\ \mathrm{x} = \{0.39506083636083 + 0.26610571912414\ \mathrm{I}\},\ \mathrm{y} = \{-5.0087312447442 - 3.799622077409\ \mathrm{I}\}$

$(\mathrm{k1} = 0),\ \mathrm{x} = \{0.40935674194543 + 0.37057376209565\ \mathrm{I}\},\ \mathrm{y} = \{1.4921022421124 - 2.8974295671847\ \mathrm{I}\}$

$(\mathrm{k1} = 1),\ \mathrm{x} = \left\{0.41396940961811 + 0.{}^{*}10^{-15}\ \mathrm{I}\right\},\ \mathrm{y} = \left\{5.7074792232379 + 0.{}^{*}10^{-14}\ \mathrm{I}\right\}$

$(\mathrm{k1} = 2),\ \mathrm{x} = \left\{0.41396940961811 + 0.{}^{*}10^{-15}\ \mathrm{I}\right\},\ \mathrm{y} = \left\{5.7074792232379 + 0.{}^{*}10^{-14}\ \mathrm{I}\right\}$

$(\mathrm{k1} = 3),\ \mathrm{x} = \left\{0.41396940961811 + 0.{}^{*}10^{-15}\ \mathrm{I}\right\},\ \mathrm{y} = \left\{5.7074792232379 + 0.{}^{*}10^{-14}\ \mathrm{I}\right\}$

$(\mathrm{k1} = 4),\ \mathrm{x} = \{0.16204333222171 - 0.02587870618716\ \mathrm{I}\},\ \mathrm{y} = \{23.106211186736 + 5.423863241858\ \mathrm{I}\}$

$(\mathrm{k1} = 5),\ \mathrm{x} = \{0.137862812901823 - 0.0192385777125\ \mathrm{I}\},\ \mathrm{y} = \{29.467018936677 + 5.879355175956\ \mathrm{I}\}$

Infinite Roots

IV.3.4. Case 4. We have 1 form depending on the basic categorization, therefore we have the form with **Exchange the initial values of variables**.

$$4.\ \begin{cases} \mathrm{Sin}(y) = u \Rightarrow y = \pi - \mathrm{ArcSin}[u] + 2k\pi \\ \mathrm{Log}(x) = v \Rightarrow x = e^{v} \\ x_{in} = y_0 \wedge y_{in} = x_0 \end{cases}$$

Result 10. For the first 11 roots with any value of $\{x, y\}$

$(\mathrm{k1} = -5),\ \mathrm{x} = \{0.84352397325401 - 0.02021128132134\ \mathrm{I}\},\ \mathrm{y} = \{-29.778254735429 + 4.080262802814\mathrm{I}\}$
$(\mathrm{k1} = -4),\ \mathrm{x} = \{0.80621595367466 - 0.03033523218618\ \mathrm{I}\},\ \mathrm{y} = \{-23.495103336024 + 3.974408951970\mathrm{I}\}$
$(\mathrm{k1} = -3),\ \mathrm{x} = \{0.74671026871713 - 0.05089579895894\ \mathrm{I}\},\ \mathrm{y} = \{-17.221081346081 + 3.868985427445\mathrm{I}\}$
$(\mathrm{k1} = -2),\ \mathrm{x} = \{0.64023848861854 - 0.10243324185352\ \mathrm{I}\},\ \mathrm{y} = \{-10.986999722396 + 3.785375163762\mathrm{I}\}$
$(\mathrm{k1} = -1),\ \mathrm{x} = \{0.39506083636083 - 0.2661057191241\ \mathrm{I}\},\ \mathrm{y} = \{-5.0087312447442 + 3.7996220774093\mathrm{I}\}$
$(\mathrm{k1} = 0),\ \mathrm{x} = \{0.40935674194543 - 0.3705737620956\ \mathrm{I}\},\ \mathrm{y} = \{1.49210224211241 + 2.8974295671847\ \mathrm{I}\}$
$(\mathrm{k1} = 1),\ \mathrm{x} = \left\{0.41396940961811 + 0. * 10^{-15}\ \mathrm{I}\right\},\ \mathrm{y} = \left\{5.7074792232379 + 0. * 10^{-14}\mathrm{I}\right\}$
$(\mathrm{k1} = 2),\ \mathrm{x} = \left\{0.41396940961811 + 0. * 10^{-15}\ \mathrm{I}\right\},\ \mathrm{y} = \left\{5.7074792232379 + 0.^{*}10^{-14}\mathrm{I}\right\}$
$(\mathrm{k1} = 3),\ \mathrm{x} = \left\{0.41396940961811 + 0. * 10^{-15}\ \mathrm{I}\right\},\ \mathrm{y} = \left\{5.7074792232379 + 00^{*}10^{-14}\mathrm{I}\right\}$
$(\mathrm{k1} = 4),\ \mathrm{x} = \{0.16204333222171 + 0.025878706187\ \mathrm{I}\},\ \mathrm{y} = \{23.106211186736 - 5.42386324185801\mathrm{I}\}$
$(\mathrm{k1} = 5),\ \mathrm{x} = \{0.137862812901823 + 0.01923857771\ \mathrm{I}\},\ \mathrm{y} = \{29.467018936677 - 5.87935517595601\ \mathrm{I}\}$
Infinite Roots

VI. 4 &5&6th Category of Roots

These 3 categories are simple and have 2 root each. Therefore, if we calculate the **Exchange** and the **initial values of variables**, the total number of roots is 12.

VI.1 4^{th} Category of Roots.

We have 1 form depending on the basic categorization.

$$\left\langle \begin{array}{l} x = u_1 \\ y = u_2 \end{array} \right.$$

That like we have the Program:

Program in mathematica..

```
Off[General::munfl,FindRoot::cvmit];
h2g2[x_, y_] := N [(5 − y* Cos[x]) / Sin[y]];
h2g1[x_, y_] := −N [(Exp [x*y] + x − 6) / log[x]];
g1[x_, y_] := N[y ∗ Cos[x] + x ∗ Sin[y] − 5];
g2[x_, y_] := N [Exp [x*y] + y* log[x] + x − 6];
c := {−2, 2}; d := 1;
For[k1 = −d, k1 <= d, k1 + +,
f1[u1_] := N[u1]
f2[u2_] := N[u2];
If [k1 < 0,Goto [mama1], Goto [mama2]];
Label [mama1];
xr1=N[Nest[f1[h2g1[#,#]]&,N[c],1],100];
yr2=N[Nest [f2[h2g2[#,#]]&,N[c],1],100];
Goto [mama3];
Label [mama2];
xr1=N[Nest[f1[h2g1[#,#]]&,N[c],1],200];
yr2=N[Nest [f2[ h2 g2[#, #]]&,  N[c], 1], 200];
Label [mama3];
FQ1 = N[xr1, 275];
FQ2 = N[yr2, 275];
s1 = N[x1, 100]/. FindRoot [{y1 ∗ Cos[x1] + x1 ∗ Sin[y1] − 5 == 0, Exp[x1 ∗ y1] + y1 ∗ log[x1] + x1 − 6 = 0};
{x1, FQ1}, {y1, FQ2}, WorkingPrecision → 550];
s2 = N[y1, 100]/ FindRoot [{y1 ∗ Cos[x1] + x1 ∗ Sin[y1] − 5 == 0, Exp[x1 ∗ y1] + y1 ∗ log[x1] + x1 − 6 == 0};
{x1, FQ1}, {y1, FQ2}, WorkingPrecision → 550];
Print [" (k1 = ", k1, ″) , ,′ x =  ", N[s1, 14], ", y = ",  N[s2, 14]]];
```

i)Result 11. For only 2 roots of $\{x, y\}$

```
(k1 = 0),  x = {−2.0621768656907 + 3.8009347638973 I},  y = {−10.5449802077644 + 4.7845252332165 I}
(k1 = 0),  x = {−58.298938832232 + 10.491648773039 I},  y = {8.9995875568204 − 8.9511928467145 I}
```

ii) Result 12. The form with Exchange the initial values of variables for $x1 = FQ2$&$y1 = FQ1$& $\{c = 1/2$&$c = 2\}$ respectively gives the values.

```
(k1 = 0),  x = {−8.6825653410131 + / − 5.5056015115416 I},  y = {−3.5801294488746 + / − 4.9913955475057 I}
(k1 = 0),  x = {0.41396940961811},  y = {5.7074792232379}
```

VI.2. 5^{th} Category of Roots.

We have 1 form depending on the basic categorization.

$$\left\langle \begin{array}{l} y = u_1 \\ x = u_2 \end{array}\right.$$

That like we have the Program:

Program in mathematica..

```
Off[General::munfl,FindRoot::cvmit];
h2g2 [x_, y_] := N[(5 − x ∗ Sin[y])/ Cos[x]];
h2g1[x_, y_] := −N [(Exp [x*y] + y* log[x] − 6)];
g1 [x_, y_] := N [y* Cos[x] + x ∗ Sin[y] − 5];
g2 [x_y_] := N [Exp [x*y] + y* log[x] + x − 6];
c := {−1/10, 1/10}; d := 1;
For [k1 = −d, k1 <= d, k1 + +;
f1[u1]:=N[u1];
f2[u2]:=N[u2];
If [k1 < 0, Goto [mama1], Goto [mama2]];
Label [mama1];
xr1=N[Nest[f1[h2g1[#,#]]&,N[c],1],100];
yr2 = N[Nest[f2[ h2 g2[#, #]]&,  N[c], 1], 100];
Goto [mama3];
Label [mama2];
xr1 = N[Nest[f1[ h2 g1[#, #]]&,  N[c], 1], 200];
yr2 = N[Nest[f2[ h2 g2[#, #]]&,  N[c], 1], 200];
Label [mama3];
FQ1 = N[xr11, 275];
FQ2 = N[yr2, 275];
s1 = N[x1, 100]/. FindRoot [{y1 ∗ Cos[x1] + x1 ∗ Sin[y1] − 5 == 0, Exp[x1 ∗ y1] + y1 ∗ log[x1] + x1 − 6 = 0 = 0};
{x1, FQ1}, {y1, FQ2}, WorkingPrecision → 550];
s2 = N[y1, 100]/ FindRoot [{y1 ∗ Cos[x1] + x1 ∗ Sin[y1] − 5 == 0, Exp[x1 ∗ y1] + y1 ∗ log[x1] + x1 − 6 == 0};
{x1, FQ1}, {y1, FQ2}, WorkingPrecision → 550];
Print [" (k1 = ", k1, ," ) , ", "x = ", , N[s1, 14], ", y = ",  N[s2, 14]]];
```

i). Result 13. For only 2 roots of $\{x, y\}$

```
(k1 = 0),  x = {4.5501190011626 − 3.3284815835014 I},  y = {0.03914909282626 + 0.28640652026055 I}
(k1 = 0),  x = {0.41396940961811},  y = {5.7074792232379}
```

ii). Result 14. The form with Exchange the initial values of variables for x1 = FQ2&$y1$ = FQ1& $\{c = -1/10\&c = 1/10\}$ respectively gives the values

```
(k1 = 0),  x = {3.3020574089305 − 2.5634842292414 I},  y = {−0.7310598576320 + 1.3510122375647 I}
(k1 = 0),  x = {0.41396940961811},  y = {5.7074792232379}
```

VI.3. 6^{th} Category of Roots.

We have 1 form depending on the basic categorization.

$$\left\langle \begin{array}{l} y = u_1 \\ \log(x) = u_2 \Rightarrow x = e^{u2} \end{array} \right.$$

That like we have the Program:

Program in mathematica.

```
Off[General::munfl, FindRoot::cvmit];
h2g1 [x_, y_] := N[(5 − x ∗ Sin[y])/ Cos[x]];
h2g2 [x_y_] := −N [(Exp [x*y] + x − 6) /y];
g1 [x_, y] := N[y ∗ Cos[x] + x ∗ Sin[y] − 5];
g2 [x_, y_] := N [Exp [x*y] + y* log[x] + x − 6];
c := {2, −1/2}; d := 1;
For [k1 = −d, k1 <= d, k1 + +,
f1[u1_] := N[u1];
f2[u2_] := N[Exp[u2]];
If [k1 < 0, Goto[mama1], Goto[mama 2]];
Label[mama1];
xr1=N[Nest [f2[ h2 g2[#, #]]&,  N[c], 1], 100];
yr2 = N[Nest[f1[ h2 g1[#, #]]&,  N[c], 1], 100];
Goto[mama3];
Label[mama2];
xr1=N[Nest [f2[ h2 g2[#, #]]&,  N[c], 1], 200];
yr2 = N[Nest[f1[ h2 g1[#, #]]&,  N[c], 1], 200];
Label[mama3];
FQ1=N[xr1,275];
FQ2 = N[yr2, 275];
s1 = N[x1, 100]/· FindRoot [{y1 ∗ Cos[x1] + x1 ∗ Sin[y1] − 5 == 0, Exp[x1 ∗ y1] + y1 ∗ log[x1] + x1 − 6 = 0},
{x1, FQ1}, {y1, FQ2}, WorkingPrecision → 550];
s2 = N[y1, 100]/· FindRoot [{y1 ∗ Cos[x1] + x1 ∗ Sin[y1] − 5 == 0, Exp[x1 ∗ y1] + y1 ∗ log[x1] + x1 − 6 = 0},
{x1, FQ1}, {y1, FQ2}, WorkingPrecision → 550];
Print[”(k1=”,k1,”)”,”, ”x=”,N[s1,14],”, y = ”, N[s2, 14]];
Print[N[g1[s1,s2],45],”,”,N[g2[s1,s2],45]]];
```

i). Result15. For only 2 roots with of $\{x, y\}$

(k1 = 0), x = {0.58589267395229 + 0.75987830580662 I}, y = {6.0322848792995 + 1.6872931285556 I}
(k1 = 0), x = {0.41396940961811}, y = {5.7074792232379}

ii). Result 16. The form with Exchange the initial values of variables for x1 = FQ2&y1 = FQ& $\{c = -1/2\&c = 2\}$ respectively gives the values.

(k1 = 0), x = {10.5826029758791 − / + 3.38986380496515 I}, y = {0.07525688652477 + / − 0.2720965330743 I}
(k1 = 0), x = {0.58589267395229 + / − 0.75987830580662 I}, y = {6.0322848792995 + / − 1.6872931285556 I}

VII. 7$^{\mathbf{th}}$ Category of Roots

We have 3 forms according to the basic categorization mentioned and we will have 6 cases in total if we add the exchange of the initial values of the variables. In this category we use Newton's form of approximation with groups **in 1 - Multiple filters**. We have 2x2 instances of multiplicity of roots k1Xk2 which gives us k2 times infinite roots of k on negative and positive level and 1x2 instances of multiplicity of roots k on negative and positive level.

$$1.\ \left\langle \begin{aligned} & x \cdot y = u \wedge y = u/x \\ & e^{u} = u_2 \Rightarrow u = \log\left(u_2\right) + 2k_2\pi i \\ & \operatorname{Cos}(x) = u_1 \Rightarrow x = \operatorname{ArcCos}\left[u_1\right] + 2k_1\pi \end{aligned} \right.$$

$$2.\ \left\langle \begin{aligned} & x \cdot y = u \wedge y = u/x \\ & e^{u} = u_2 \Rightarrow u = \log\left(u_2\right) + 2k_2\pi i \\ & \operatorname{Cos}(x) = u_1 \Rightarrow x = -\operatorname{ArcCos}\left[u_1\right] + 2k_1\pi \end{aligned} \right.$$

$$3.\ \begin{cases} x \cdot y = u \wedge y = u/x \\ e^{u} = u_2 \Rightarrow u = \log\left(u_2\right) + 2k\pi i \\ x = u_1 \end{cases}$$

With the method we use, we exploit the complex function $e^{x \cdot y} = u \Rightarrow x \cdot y = \log(u) + 2k\pi i$ so that we can use it to get a new multiplicity of roots and not lose. This is a necessity in the root finding process.

VII.1. Case 1. We have 1 form depending on the basic categorization, therefore we have the form with **normal initial values** of the variables

$$1.\ \begin{cases} x \cdot y = u \wedge y = u/x \\ e^{u} = u_2 \Rightarrow u = \log\left(u_2\right) + 2k_2\pi i \\ \text{Cos}(x) = u_1 \Rightarrow x = \text{ArcCos}\left[u_1\right] + 2k_1\pi \end{cases}$$

That like we have the Program:

Program in mathematica.

```
Off[General::munfl,FindRoot::cvmit];
h2g1 [x_, u_] := N[(5 − x ∗ Sin[u/x])x/u];
h2g2[x_, u_] := −N[(u/x ∗ log[x] + x − 6)];
g1 [x_, u_] := N[u/x ∗ Cos[x] + x ∗ Sin[u/x] − 5];
g2[x_, u_] := N[Exp[u] + u/x ∗ log[x] + x − 6];
g11[x_, y_] := N[y ∗ Cos[x] + x ∗ Sin[y] − 5];
g22[x_, y_] := N [Exp [x*y] + y* log[x] + x − 6];
c := {−0.00001}; d := 10; k2 := 0;
For [k1 = −d, k1 <= d, k1 + +,
f1[u1_] := N[ArcCos[u1] + 2πk1];
f2[u2_] := N [log[u2] + 2*k2*I*π];
If [k1 < 0, Goto [mama1], Goto [mama2]];
Label[mama1];
xr1 = N[Nest[f1[ h2 g1[#, #]]&,  N[c], 4], 100];
yr2=N[Nest [f2[ h2 g2[#, #]]&,  N[c], 4], 100];
Goto[mama3];
Label[mama2];
xr1=N[Nest[f1[h2g1[#,#]]&,N[c],2],100];
yr2=N[Nest[f2[h2g2[#,#]]&,N[c],2],100];
Label[mama3];
FQ1 = N[xr1, 475];
FQ2 = N[yr2, 475];
s1 = N[x1, 100]/.FindRoot[{u1/x1∗Cos[x1]+x1∗Sin[u1/x1]−5 == 0, Exp[u1]+u1/x1∗log[x1]+x1−6 == 0},
{x1, FQ1}, {u1, FQ2}, WorkingPrecision → 450];
s2 = N[u1, 100]/. FindRoot[{u1/x1∗Cos[x1]+x1∗Sin[u1/x1]−5 == 0, Exp[u1]+u1/x1∗log[x1]+x1−6 = 0},
{x1, FQ1}, {u1, FQ2}, WorkingPrecision → 450];
```

Result 17. For the first 11 roots with any value of $\{x, y\}$

$(\mathrm{k1} = -5),\ \mathrm{x} = \{-40.832043798563 + 3.195732428237\ \mathrm{I}\},\ \mathrm{y} = \{3.8554774222462 - 0.0611186112702\ \mathrm{I}\}$
$(\mathrm{k1} = -4),\ \mathrm{x} = \{-28.240209221193 - 3.1526908003801\ \mathrm{I}\},\ \mathrm{y} = \{3.5476341337688 + 0.0789344538321\ \mathrm{I}\}$
$(\mathrm{k1} = -3),\ \mathrm{x} = \{-28.240209221193 - 3.152690800380\ \mathrm{I}\},\ \mathrm{y} = \{3.5476341337688 + 0.0789344538328\ \mathrm{I}\}$
$(\mathrm{k1} = -2),\ \mathrm{x} = \{-15.610260481764 - 2.978644522260\ \mathrm{I}\},\ \mathrm{y} = \{3.0990587880259 + 0.1050851183124\ \mathrm{I}\}$
$(\mathrm{k1} = -1),\ \mathrm{x} = \{-15.610260481764 - 2.978644522260\ \mathrm{I}\},\ \mathrm{y} = \{3.0990587880259 + 0.1050851183124\ \mathrm{I}\}$
$(\mathrm{k1} = 0),\ \mathrm{x} = \{-0.81207408344963 + 0.08955624153\ \mathrm{I}\},\ \mathrm{y} = \{1.1548147085202 + 2.0683154790\ \mathrm{I}\}$
$(\mathrm{k1} = 1),\ \mathrm{x} = \{4.55011900116261 + 3.328481583501\ \mathrm{I}\},\ \mathrm{y} = \{1.1314318592290 - 1.1728767154114\ \mathrm{I}\}$
$(\mathrm{k1} = 2),\ \mathrm{x} = \{-2.5065032563201 + 2.239559090232\ \mathrm{I}\},\ \mathrm{y} = \{2.0975869781721 - 0.0734136595365\ \mathrm{I}\}$
$(\mathrm{k1} = 3),\ \mathrm{x} = \{-9.2319815063655 + 2.763894181135\ \mathrm{I}\},\ \mathrm{y} = \{2.7577350323453 - 0.1152391309089\ \mathrm{I}\}$
$(\mathrm{k1} = 4),\ \mathrm{x} = \{-2.5065032563201 + 2.239559090232\ \mathrm{I}\},\ \mathrm{y} = \{2.0975869781721 - 0.0734136595365\ \mathrm{I}\}$
$(\mathrm{k1} = 5),\ \mathrm{x} = \{-2.5065032563201 + 2.239559090232\ \mathrm{I}\},\ \mathrm{y} = \{2.0975869781721 - 0.0734136595365\ \mathrm{I}\}$
Infinite Roots

VII.2. Case 2. We have 1 form depending on the basic categorization , therefore we have the form with **normal initial values** of the variables

$$2.\ \begin{cases} x \cdot y = u \wedge y = u/x \\ e^{u} = u_2 \Rightarrow u = \log\left(u_2\right) + 2k_2\pi i \\ \mathrm{Cos}(x) = u_1 \Rightarrow x = -\,\mathrm{ArcCos}\left[u_1\right] + 2k_1\pi \end{cases}$$

That like we have the Program:

Program in mathematica.

```
Off[General::munfl,FindRoot::cvmit];
h2g1 [x_, u_] := N [(5 − x* Sin[u/x]) x/u];
h2g2 [x_, u_] := −N[(u/x ∗ log[x] + x − 6)];
g1 [x_, u_] = N[u/x ∗ Cos[x] + x ∗ Sin[u/x] − 5]; g2 [x_; u_] := N[Exp[u] + u/x ∗ log[x] + x − 6];
g11 [x_, y_] = N [y* Cos[x] + x ∗ Sin[y] − 5];
g22 [x_, y_] = N [Exp [x*y] + y* log[x] + x − 6];
c := {−0.00001}; d := 10; k2 := 0;
For [kl = −d, kl <= d, k1 + +,
f1[u1_] = N[− ArcCos[u1] + 2πk1];
f2[u2_] = N [log[u2] + 2 ∗ k2 ∗ I*π];
If [k1 < 0, Goto [mama1],Goto [mama2]];
Label[mama1];
xr1 = N[Nest[f1[h2g1[#, #]]&, N[c], 4], 200];
yr2 = N[Nest[f2[h2g2[#, #]]&, N[c], 4], 200];
Goto[mama3];
Label[mama2];
xr1 = N[Nest[fl[h2 g1[#, #]]&,  N[c], 2], 200];
yr2 = N[Nest[f2[ h2 g2[#, #]]&,  N[c], 2], 200];
Label[mama3];
FQ1 = N[xr1, 475]; FQ2 = N[yr2, 475];
s1 = N[x1, 100]/.FindRoot[{u1/x1* Cos[x1] + x1 ∗ Sin[u1/x1] − 5 == 0, Exp[u1] + u1/x1 ∗ log[x1] + x1 − 6 == 0} ,
{x1, FQ1}, {u1, FQ2}, WorkingPrecision → 550];
s2 =N[u1,100]/.FindRoot [{u1/x1∗Cos[x1]+x1∗Sin[u1/x1]−5 == 0, Exp[u1]+u1/x1∗log[x1]+x1−6 == 0},
{x1, FQ1}, {u1, FQ2}, WorkingPrecision→ 550]; Print[”(k1=”,k1,”),”,”x=”,N[s1,14],”,y=”,N[s2,14]]];
```

Result 18. For the first 11 roots with any value of $\{x, y\}$

```
(k1 = −5), x = {−47.123527978564 − 3.191048395199 I}, y = {3.9799323537023 + 0.0544967566087 I}
(k1 = −4), x = {−40.832043798563 − 3.195732428237 I}, y = {3.8554774222462 + 0.0611186112702 I}
(k1 = −3), x = {−28.240209221193 + 3.152690800380 I}, y = {3.5476341337688 − 0.0789344538328 I}
(k1 = −2), x = {−21.934242405453 − 3.090694054749 I}, y = {3.3487261595698 + 0.0910006971651 I}
(k1 = −1), x = {−9.2319815063655 − 2.763894181135 I}, y = {2.7577350323453 + 0.1152391309089 I}
(k1 = 0), x = {−2.5065032563201 + 2.239559090232 I}, y = {2.0975869781721 − 0.0734136595365 I}
(k1 = 1), x = {4.5501190011626 + 3.3284815835014 I}, y = {1.1314318592290 − 1.1728767154114 I}
(k1 = 2), x = {4.5501190011626 + 3.3284815835014 I}, y = {1.1314318592290 − 1.1728767154114 I}
(k1 = 3), x = {4.5501190011626 + 3.3284815835014 I}, y = {1.1314318592290 − 1.1728767154114 I}
(k1 = 4), x = {10.5826029758791 − 3.389863804965 I}, y = {1.7187839402178 + 2.6243789849345 I}
(k1 = 5), x = {4.5501190011626 + 3.3284815835014 I}, y = {1.1314318592290 − 1.1728767154114 I}
Infinite Roots
```

VII.3. Case 3. We have 1 form depending on the basic categorization, therefore we have the form with **normal initial values** of the variables

$$3.\ \begin{cases} x \cdot y = u \wedge y = u/x \\ e^{u} = u_2 \Rightarrow u = \log\left(u_2\right) + 2k\pi i \\ x = u_1 \end{cases}$$

```
Off[General::munfl,FindRoot::cvmit];
h2g1 [y_u_] := N[(5 − u/y ∗ Sin[y])/ Cos[u/y]];
h2g2 [y_, u_] := −N [(y* log[u/y] + u/y − 6)];
g1 [y_, u_] := N [y* Cos[u/y] + u/y* Sin[y] − 5];
g2 [y_, u_] := N [Exp[u] + y* log[u/y] + u/y − 6];
g11[x_, y_] := N[y ∗ Cos[x] + x ∗ Sin[y] − 5];
g22 [x_y_] := N [Exp [x*y] + y* log[x] + x − 6];
c := {−0.00001}; d := 10;
For [k = −d, k <= d, k + +,
f1[u1_] := N[u1];
f2[u2_] := N [log[u2] + 2 ∗ k*I*π];
If [kl < 0, Goto [mamal ], Goto [mama2]];
Label[mama1];
xr1=N[Nest[f1[h2g1[#,#]]&,N[c],2],100];
yr2=N[Nest[f2[h2g2[#,#]]&,N[c],2],100];
Goto[mama3];
Label[mama2];
xr1=N[Nest[f1[h2g1[#,#]]&,N[c],2],100];
yr2=N[Nest[f2[h2g2[#,#]]&,N[c],2],100];
Label[mama3];
FQ1=N[xr1,475];
FQ2=N[yr2,475];
s1=N[y1,100]/.FindRoot[{y1∗Cos[u1/y1]+u1/y1∗Sin[y1]−5 == 0, Exp[u1]+y1∗log[u1/y1]+u1/y1−6 ==
0}, {y1, FQ1}, {u1, FQ2}, WorkingPrecision→ 650];
s2=N[u1,,100]/.FindRoot [{y1∗Cos[u1/y1]+u1/y1∗Sin[y1]−5 == 0, Exp[u1]+y1∗log[u1/y1]+u1/y1−6 ==
0}, {y1, FQ1}, {u1, FQ2}, WorkingPrecision→ 650];
Print[”(k=”,k,”),”,”x=”,N[s1,14],”,y=”,N[s2,14]]]
```

Result 19. For the first 11 roots with any value of $\{x, y\}$

```
(k = −5), x = {−0.4799021739488 + 0.399503018401 I}, y = {2.0975869781721 + 0.0734136595365 I}
(k = −4), x = {12.4440052943089 − 3.3021215482000 I}, y = {2.858743476125 − 17.58116545610300 I}
(k = −3), x = {9.0017227411180 + 3.0212080997081 I}, y = {2.7418767649514 − 10.9659654793819I}
(k = −2), x = {9.0017227411180 + 3.0212080997081 I}, y = {2.7418767649514 − 10.9659654793819 I}
(k = −1), x = {8.9188085102266 + 3.1040817140224I}, y = {2.7614567724023 − 5.0934634431111 I}
(k = 0), x = {5.7074792232379}, y = {2.3627218044514}
(k = 1), x = {8.9188085102266 − 3.1040817140224 I}, y = {2.7614567724023 + 5.0934634431111 I}
(k = 2), x = {9.0017227411180 − 3.0212080997081I}, y = {2.7418767649514 + 10.9659654793819 I}
(k = 3), x = {9.0017227411180 − 3.0212080997081 I}, y = {2.7418767649514 + 10.9659654793819 I}
(k = 4), x = {12.4440052943089 + 3.302121548200 I}, y = {2.8587434761251 + 17.5811654561031 I}
(k = 5), x = {−0.47990217394887 − 0.399503018409 I}, y = {2.09758697817210 − 0.07341365953650 I}
```

Infinite Roots

VII.4.1. Case 1. We have 1 form depending on the basic categorization, therefore we have the form with **exchange of the initial values** of the variables

$$1.\ \begin{cases} x \cdot y = u \wedge y = u/x \\ e^{u} = u_2 \Rightarrow u = \log\left(u_2\right) + 2k_2\pi i \\ \operatorname{Cos}(x) = u_1 \Rightarrow x = \operatorname{ArcCos}\left[u_1\right] + 2k_1\pi \end{cases}$$

That like we have the Program:

Program in mathematica.

```
Off[General::munfl, FindRoot::cvmit];
h2g1 [x_, u_] = N [(5 − x* Sin[u/x]) x/u];
h2g2 [x_, u] ] := −N[(u/x ∗ Lox[x] + x − 6)];
g1 [x_, u_] := N [u/x* Cos[x] + x ∗ Sin[u/x] − 5];
g2 [x_u_] := N[Exp[u] + u/x ∗ log[x] + x − 6];
g11 [x_, y_] := N[y ∗ Cos[x] + x ∗ Sin[y] − 5];
g22 [x_, y_] := N [Exp[x ∗ y] + y* log[x] + x − 6];
c := {−0.00001}; d := 10; k2 := 0;
For [k1 = −d, k1 <= d, k1 + +,
f1 [u1_] := N[ArcCos[u1] + 2πk1];
```

```
f2 [u2_] := N [log[u2] + 2 ∗ k2*I*π];
If [k1 < 0, Goto[mama1], Goto[mama2]];Label[mama1];
xr1 = N[Nest[f1[ h2 g1[#, #]]&,  N[c], 4], 100];
yr2 = N[Nest[f2[ h2 g2[#, #]]&,  N[c], 4], 100];
Goto [mama 3]; Label [mama 2];
xr1 = N[Nest[f1[ h2 g1[#, #]]&,  N[c], 4], 100];
yr2 = N[Nest[f2[ h2 g2[#, #]]&,  N[c], 4], 100]; Label[mama3];
FQ1 = N[xr1, 475];
FQ2 = N[yr2, 475];
s1 = N[x1, 100]/FindRoot[{u1/x1∗Cos[x1]+x1∗Sin[u1/x1]−5 == 0, Exp[u1]+u1/x1∗log[x1]+x1−6 == 0},
{x1, FQ2}, {u1, FQ1}, WorkingPrecision→ 450];
s2 = N[u1, 100]/FindRoot[{u1/x1∗Cos[x1]+x1∗Sin[u1/x1]−5 == 0, Exp[u1]+u1/x1∗log[x1]+x1−6 == 0},
{x1, FQ2}, {u1, FQ1}, WorkingPrecision→ 450];
Print[”(kl=”,  k1, ”), ”, x′′, N[s1, 14], ”, y = ”,  N[ s2, 14]]]
```

Result 20. For the first 11 roots with any value of $\{x, y\}$

```
(k1 = −5),  x = {0.80621595367466 − 0.03033523218618 I},  y = {−18.821562524376 + 3.916961318442 I}
(k1 = −4),  x = {0.74671026871713 − 0.05089579895894 I},  y = {−12.6622431750410 + 3.76549184223 I}
(k1 = −3),  x = {0.39506083636083 − 0.26610571912414 I},  y = {−0.9676523893463 + 2.833933905538 I}
(k1 = −2),  x = {0.64023848861854 − 0.10243324185352 I},  y = {−6.6465518470635 + 3.548976873510 I}
(k1 = −1),  x = {0.39506083636083 − 0.26610571912414 I},  y = {−0.9676523893463 + 2.833933905538 I}
(k1 = 0),  x = {0.40935674194543 − 0.37057376209565 I},  y = {1.6845134875994 + 0.6331483863481 I}
(k1 = 1),  x = {0.41396940961811},  y = {2.3627218044514}
(k1 = 2),  x = {0.26968386392917 − 0.06278973577459 I}, y = {2.9559053239136 + 0.3848869038066 I}
(k1 = 3),  x = {0.25337506130069 + 0.26983741589744 I},  y = {3.3678274560011 + 5.9093899411254 I}
(k1 = 4),  x = {0.16244443201109 + 0.16761865292665 I},  y = {4.0431168944285 + 5.9857183414399 I}
(k1 = 5),  x = {0.11622481581891 + 0.36979731407009 I},  y = {3.3519153490553 + 5.2813686046948 I}
```

Infinite Roots

VII.4.2. Case 2. We have 1 form depending on the basic categorization, therefore we have the form with **exchange of the initial values** of the variables

$$2. \begin{cases} x \cdot y = u \wedge y = u/x \\ e^{u} = u_2 \Rightarrow u = \log\left(u_2\right) + 2k_2\pi i \\ \operatorname{Cos}(x) = u_1 \Rightarrow x = -\operatorname{ArcCos}\left[u_1\right] + 2k_1\pi \end{cases}$$

That like we have the Program:

Program in mathematica.

```
Off[General::munfl,FindRoot::cvmit];
h2g1 [x_, u_] := N [(5 − x* Sin[u/x]) x/u];
h2g2 [x_, u_] := −N [(u/x* log[x] + x − 6)];
g1 [x_, u_] := N [u/x* Cos[x] + x* Sin[u/x] − 5];
```

```
g2[x_, u_] := N[Exp[u] + u/x ∗ log[x] + x − 6];
g11 [x_, y_] := N [y* Cos[x] + x ∗ Sin[y] − 5];
g22 [x_, y_] := N [Exp [x*y] + y* log[x] + x − 6];
c := {−0.00001}; d := 10; k2 := 0;
For [k1 = −d, k1 <= d, k1 + +,
f1[u1_] := N[− ArcCos[u1] + 2πk1];
f2[u2_] := N [log[u2] + 2 ∗ k2 ∗ I*π];
If [k1 < 0, Goto [mama 1], Goto [mama2]]; Label [mama1];
xr1 = N[Nest[f1[ h2 g1[#, #]]&,  N[c], 4], 100];
yr2 = N[Nest[f2[ h2 g2[#, #]]&,  N[c], 4], 100];
Goto [mama3]; Label [mama2];
xr1 = N[Nest[f1[ h2 g1[#, #]]&,  N[c], 2], 100];
yr2 = N [ Nest [f2[ h2 g2[#, #]]&,  N[c], 2], 100]; Label [mama3];
FQ1 = N[xr11, 475];
FQ2 = N[yr2, 475];
s1 = N[x1, 100]/ FindRoot [{u1/x1∗Cos[x1]+x1∗Sin[u1/x1]−5 == 0, Exp[u1]+u1/x1∗log[x1]+x1−6 == 0},
{x1, FQ2}, {u1, FQ1}, WorkingPrecision→ 450];
s2 = N[u1, 100]/ FindRoot [{u1/x1∗Cos[x1]+x1∗Sin[u1/x1]−5 == 0, Exp[u1]+u1/x1∗log[x1]+x1−6 == 0},
{x1, FQ2}, {u1, FQ1}, WorkingPrecision→ 450];
Print[”(k1=”,k1,”)”,”,” x = ”, N[s1, 14], ”, y = ”, N[s2, 14]]]
```

Result 21. For the first 11 roots with any value of $\{x, y\}$

```
(k1 = −5),  x = {0.80621595367466 − 0.03033523218618 I},  y = {−18.821562524376 + 3.916961318442 I}
(k1 = −4),  x = {0.74671026871713 − 0.05089579895894 I},  y = {−12.6622431750410 + 3.76549184223 I}
(k1 = −3),  x = {−2.5065032563201 − 2.239559090232501 I},  y = {2.09758697817210 + 0.073413659536 I}
(k1 = −2),  x = {0.64023848861854 − 0.10243324185352 I},  y = {−6.6465518470635 + 3.548976873510 I}
(k1 = −1),  x = {0.39506083636083 − 0.26610571912414 I},  y = {−0.9676523893463 + 2.833933905538 I}
(k1 = 0),  x = {−0.81207408344963 − 0.08955624153056 I},  y = {1.1548147085202 − 2.0683154790342 I}
(k1 = 1),  x = {0.41396940961811},  y = {2.3627218044514}
(k1 = 2),  x = {0.58589267395229 + 0.75987830580662 I},  y = {2.2521340740487 + 5.5723750970558 I}
(k1 = 3),  x = {0.58589267395229 + 0.75987830580662 I},  y = {2.2521340740487 + 5.5723750970558 I}
(k1 = 4),  x = {0.11622481581891 + 0.36979731407009 I},  y = {3.3519153490553 + 5.2813686046948 I}
(k1 = 5),  x = {0.16244443201109 + 0.16761865292665 I},  y = {4.0431168944285 + 5.9857183414399 I}
```

Infinite Roots

VII.4.3. Case 3. We have 1 form depending on the basic categorization, therefore we have the form with **exchange of the initial values** of the variables

$$3.\ \begin{cases} x \cdot y = u \wedge y = u/x \\ e^{u} = u_2 \Rightarrow u = \log\left(u_2\right) + 2k\pi i \\ x = u_1 \end{cases}$$

That like we have the Program:
Program in mathematica.

```
Off[General::munfl,FindRoot::cvmit];
h2g1[y_, u_] := N [(5 − u/y* Sin[y]) / Cos[u/y]];
h2g2 [y_, u_] := −N [(y* log[u/y] + u/y − 6)];
g1 [y_, u_] := N [y* Cos[u/y] + u/y* Sin[y] − 5];
g11[x_, y_] := N[Exp[u] +y ∗ Log[u/y] + u/y-6];
g22[x_, y_] := N [Exp [x*y] + y* log[x] + x − 6];
c := {−0.00001}; d := 10;
For [k = −d, k <= d, k + +,
f1[u1_]:= N[u1];
f2[u2_] := N [log[u2] + 2 ∗ k*I*π];
If [k1 < 0, Goto [mama1], Goto [mama2]];
Label[mama1];
xr1=N[Nest[f1[h2g1[#, #]]&, N[c], 2], 100];
yr2=N[Nest[f2[h2g2[#,#]]&,N[c],2],100];
Goto[mama3];
Label[mama2];
xr1=N[Nest[f1[h2g1[#,#]]&,N[c],2],100];
yr2=N[Nest[f2[h2g2[#,#]]&,N[c],2],100];
Label[mama3];
FQ1=N[xr1,475];
FQ2=N[yr2,475];
s1 = N[y1, 100]/FindRoot[{y1∗Cos[u1/y1]+u1/y1∗Sin[y1]−5 == 0, Exp[u1]+y1∗log[u1/y1]+u1/y1−6 ==
0},{y1, FQ2}, {u1, FQ1}, WorkingPrecision→ 650];
s2 = N[u1, 100]/FindRoot [{y1∗Cos[u1/y1]+u1/y1∗Sin[y1]−5 == 0, Exp[u1]+y1∗log[u1/y1]+u1/y1−6 ==
0},{y1, FQ2}, {u1, FQ1}, WorkingPrecision→ 650];
Print[”(k = ”, k, ”), ”, ”x = ”, N[s1, 14], ”, y = ”, N[s2, 14]]]
```

Result 22. For the first 11 roots with any value of $\{x, y\}$

$(\mathrm{k} = -1),\ \mathrm{x} = \{1.4921022421124 - 2.8974295671847\ \mathrm{I}\},\ \mathrm{y} = \{1.6845134875994 - 0.6331483863481\ \mathrm{I}\}$

$(\mathrm{k} = 0),\ \mathrm{x} = \{-0.4799021739488 + 0.3995030184090\ \mathrm{I}\},\ \mathrm{y} = \{2.0975869781721 + 0.0734136595365\ \mathrm{I}\}$

$(\mathrm{k} = 1),\ \mathrm{x} = \{1.4921022421124 + 2.8974295671847\ \mathrm{I}\},\ \mathrm{y} = \{1.6845134875994 + 0.6331483863481\ \mathrm{I}\}$

VIII. 8^{th} Category of Roots

We have 3 forms according to the basic categorization (normal **of the initial values**) mentioned and we will have 6 cases in total if we add the **exchange of the initial values** of the variables. In this category we use Newton's form of approximation with groups **in 1 - Multiple filters**. We have 2×2 instances of multiplicity of roots k1Xk2 which gives us k2 times infinite roots of k on negative and positive level and 1x2 instances of multiplicity of roots k on negative and positive level.

$$1. \left\langle \begin{aligned} & x \cdot y = u \wedge x = u/y \\ & e^{u} = u_2 \Rightarrow u = \log\left(u_2\right) + 2k_2\pi i \\ & \mathrm{Sin}(y) = u_1 \Rightarrow x = \mathrm{ArcSin}\left[u_1\right] + 2k_1\pi \end{aligned} \right.$$

$$2.\ \begin{cases} x \cdot y = u \wedge x = u/y \\ e^u = u_2 \Rightarrow u = \log(u_2) + 2k_2\pi i \\ \mathrm{Sin}(y) = u_1 \Rightarrow y = \pi - \mathrm{ArcCos}[u_1] + 2k_1\pi \end{cases}$$

$$3.\ \begin{cases} x \cdot y = u \wedge x = u/y \\ e^u = u_2 \Rightarrow u = \log(u_2) + 2k\pi i \\ y = u_1 \end{cases}$$

With the method we use, we exploit the complex function

$e^{x \cdot y} = u \Rightarrow x \cdot y = \log(u) + 2k\pi i$
$x = (\log(u) + 2k\pi i)/y$

so that we can use it to get a new multiplicity of roots and not lose. This is a necessity in the root finding process.

VIII.1. Case 1. We have 1 form depending on the basic categorization, therefore we have the form with **exchange of the initial values** of the variables

$$1.\ \begin{cases} x \cdot y = u \wedge x = u/y \\ e^u = u_2 \Rightarrow u = \log(u_2) + 2k_2\pi i \\ \mathrm{Sin}(y) = u_1 \Rightarrow y = \pi - \mathrm{ArcSin}[u_1] + 2k_1\pi \end{cases}$$

That like we have the Program **in mathematica** (1 category for k2=0)

```
h2g1 [y_, u_] := N [(5 − y* Cos[u/y]) /(u/y)];
h2g2 [y_, u_] := −N [(y* log[u/y] + u/y − 6)];
g1 [y_, u_] := N [y* Cos[u/y] + u/y ∗ Sin[y] − 5];
g2 [y_, u_] := N [Exp[u] + y* log[u/y] + u/y − 6];
g11 [x_, y_] := N[y ∗ Cos[x] + x ∗ Sin[y] − 5];
g22 [x_, y_] := N [Exp [x*y] + y* log[x] + x − 6];
c := {−0.00001}; d := 10; k2 := 0;
For [k1 = −d, k1 <= d, kl + +,
f1[u1_]:=N[ArcSin[u1]+2 πk1];
f2[u2_] := N [log[u2] + 2*k2 ∗ I*π];
If [k1 < 0, Goto [mama1], Goto [mama2]];
Label [mama1];
yr1=N[Nest[f1[h2g1[#,#]]&,N[c],4],100];
ur2=N[Nest[f2[h2g2[#,#]]&,N[c],4],100];
Goto [mama3];
Label [mama2];
yr1=N[Nest[f1[h2g1[#,#]]&,N[c],3],100];
ur2 = N[Nest[f2[ h2 g2[#, #]]&,  N[c], 3], 100];
Label [mama3];
FQ1=N[yr1,475];
FQ2=N[ur2,475];
s1 = N[y1, 100]/FindRoot[{y1∗Cos[u1/y1]+u1/y1∗Sin[y1]−5 == 0, Exp[u1]+y1∗log[u1/y1]+u1/y1−6 ==
0}, {y1, FQ1}, {u1, FQ2},WorkingPrecision→ 550];
s2 = N[u1, 100]/· FindRoot[{y1∗Cos[u1/y1]+u1/y1∗Sin[y1]−5 == 0, Exp[u1]+y1∗log[u1/y1]+u1/y1−6 ==
0}, {y1, FQ1}, {u1, FQ2},WorkingPrecision→ 550];
Print [”(”, k1, ”)′′, ”x = ”, N[s2/s1, 14], ”y = ”, N[s1, 14]]]
```

Result 23. For the first 11 roots with any value of {*x*, *y*}

(−5)x = {−21.728250369528 + 6.825526642367 I}, y = {3.7769452052000 − 5.61210051636341 I}
(−4)x = {−14.294551380190 + 5.810131567768 I}, y = {2.6207715786002 − 4.76251582723871 I}
(−3)x = {−0.81207408344963 − 0.089556241530 I}, y = {−1.1274634548844 + 2.6712918349786 I}
(−2)x = {−6.5029285915811 + 3.461420432251 I}, y = {0.99828215373961 − 2.3168934679397 I}
(−1)x = {0.40935674194543 + 0.370573762095 I}, y = {1.49210224211241 − 2.8974295671847 I}
(0)*x* = {0.41396940961811}, y = {5.7074792232379}
(1)x = {4.5501190011626 + 3.3284815835014 I}, y = {0.03914909282626 − 0.28640652026055I}
(2)x = {10.5826029758791 + 3.3898638049655 I}, y = {0.07525688652477 − 0.27209653307437 I}
(3)x = {8.0312798216612 − 3.01226091169341 I}, y = {0.33917834558378 − 0.38680749334890 I}
(4)*x* = {14.257007508475 − 3.4284905839671 I}, y = {0.20720485273225 − 0.20065985069048 I}
(5)x = {8.0312798216612 − 3.01226091169324 I}, y = {0.33917834558378 − 0.38680749334890 I}
(6)x = {20.553757026449 − 3.6562934894671 I}, y = {0.15803689882539 − 0.13771568061298 I}
Infinite Roots

ii). The form with **Normal initial values of variables** for **x1 = FQ1&y1 = FQ2&{c = −1/1000}** with same program respectively we have the values.

Result 24. For the first 11 roots with any value of {*x*, *y*}

(−5)x = {2.3804466659683 − 1.2531546077541 I}, y = {1.4351201426456 − 1.5201456028492 I}
(−4)x = {2.3804466659683 − 1.2531546077543 I}, y = {1.4351201426456 − 1.5201456028492 I}
(−3)x = {−0.370580405467 − 1.7659190691386 I}, y = {2.5592862426975 + 1.5340039602620 I}
(−2)x = {0.64023848861854 + 0.102433241853 I}, y = {−10.986999722396 − 3.7853751637620 I}
(−1)x = {0.39506083636083 − 0.266105719124 I}, y = {−5.0087312447442 + 3.7996220774090 I}
(0)x = {0.40935674194543 + 0.370573762095 I}, y = {1.49210224211241 − 2.8974295671847 I}
(1)*x* = {0.41396940961811}, y = {5.7074792232379}
(2)*x* = {0.41396940961811}, y = {5.7074792232379}
(3)*x* = {0.41396940961811}, y = {5.7074792232379}
(4)x = {0.26968386392917 + 0.062789735774 I}, y = {10.0818245704855 − 3.7745009652506 I}
(5)x = {0.12062284188681 + 0.015015664910 I}, y = {35.802254645046 − 6.23999599142401 I}
Infinite Roots

VII.2. Case 2.

i) We have 1 form depending on the basic categorization, therefore we have the form with **exchange of the initial values** of the variables

$$2.\begin{cases} x \cdot y = u \wedge x = u/y \\ e^{u} = u_2 \Rightarrow u = \log\left(u_2\right) + 2k_2\pi i \\ \operatorname{Sin}(y) = u_1 \Rightarrow y = \operatorname{ArcSin}\left[u_1\right] + 2k_1\pi \end{cases}$$

That like we have the Program **in mathematica**.

```
h2g1 [y_, u_] := N [(5 − y* Cos[u/y]) /(u/y)];
h2g2 [y_, u_] := −N [(y* log[u/y] + u/y − 6)];
g1 [y_, u_] := N [y ∗ Cos[u/y] + u/y* Sin[y] − 5];
g2 [y_, u_] := N [Exp[u] + y* log[u/y] + u/y − 6];
g11[x_, y_]:=N[y*Cos  [x] + x* Sin[y] − 5];
g22 [x_, y_] :=N[Exp [x*y] + y* log[x] + x − 6];
c := {−0.00001}; d := 10; k2 := 0;
For [k1 = −d, k1 <= d, k1 + +,
f1[u1_] := N[ArcSin[u1] + 2πk1];
f2[u2_]:=N[Log[u2]  +2 ∗ k2 ∗ I*π];
If [kl < 0, Goto [mamal], Goto [mama2]];
Label [mama1];
yr1=N[Nest [f1[ h2 g1[#, #]]&,  N[c], 4], 100];
ur2 = N[Nest[f2[h2g2[#, #]]&,  N[c], 4], 100];
Goto [mama3];
Label [mama2];
yrl = N[Nest[f1[h2g1[#, #]]&, N[c], 3], 100];
ur2 = N[Nest[f2[h2g2[#, #]]&, N[c], 3], 100];
Label [mama3];
FQ1=N[yr1,475];
FQ2=N[ur2,475];
s1 = N[y1, 100]/FindRoot [{y1∗Cos[u1/y1]+u1/y1∗Sin[y1]−5 == 0, Exp[u1]+y1∗log[u1/y1]+u1/y1−6 ==
0}, {y1, FQ2}, {u1, FQ1}, WorkingPrecision  → 550];
s2 = N[u1, 100]/FindRoot[{y1∗Cos[u1/y1]+u1/y1∗Sin[y1]−5 == 0, Exp[u1]+y1∗log[u1/y1]+u1/y1−6 ==
0} {y1, FQ2}, {u1, FQ1}, WorkingPrecision  → 550];
 Print["(",k1,")","x=",N[s2/s1,14],"y=",N[s1,14]]]
```

Result 25. For the first 11 roots with any value of $\{x, y\}$

```
(−5)x = {−21.728250369528 + 6.825526642367 I},  y = {3.7769452052000 – 5.6121005163634 I}
(−4)x = {−14.294551380190 + 5.810131567768 I},  y = {2.6207715786002 – 4.7625158272387 I}
(−3)x = {−0.81207408344963 – 0.089556241530 I},  y = {−1.1274634548844 + 2.671291834978 I}
(−2)x = {−6.50292859158111 + 3.46142043225 I},  y = {0.9982821537396 – 2.3168934679397 I}
(−1)x = {0.40935674194543 + 0.370573762095 I},  y = {1.4921022421124 – 2.8974295671847 I}
(0)x = {0.41396940961811},  y = {5.7074792232379}
(1)x = {4.5501190011626 + 3.32848158350141 I},  y = {0.0391490928262 – 0.28640652026055 I}
(2)x = {10.5826029758791 + 3.3898638049655 I},  y = {0.0752568865247 – 0.27209653307437 I}
```

(3)x = {8.0312798216612 − 3.01226091169341 I}, y = {0.3391783455837 − 0.38680749334890 I}
(4)x = {14.257007508475 − 3.42849058396722 I}, y = {0.2072048527322 − 0.20065985069048 I}
(5)x = {8.0312798216612 − 3.01226091169341 I}, y = {0.3391783455837 − 0.38680749334890 I}
Infinite Roots

ii). The form with Normal initial values of variables for **x1 = FQ1&y1 = FQ2&{c = −1/1000}** with same program respectively we have the values.

Result 26. For the first 11 roots with any value of $\{x, y\}$

(−5)x = {2.3804466659683 − 1.2531546077543 I}, y = {1.4351201426456 − 1.5201456028492 I}
(−4)x = {2.3804466659683 − 1.2531546077543 I}, y = {1.4351201426456 − 1.5201456028492 I}
(−3)x = {−0.3705804054675 − 1.7659190691386 I}, y = {2.5592862426975 + 1.5340039602620 I}
(−2)x = {0.64023848861854 + 0.102433241853 I}, y = {−10.9869997223966 − 3.785375163762 I}
(−1)x = {0.39506083636083 − 0.266105719124 I}, y = {−5.00873124474421 + 3.799622077409 I}
(0)x = {0.40935674194543 + 0.3705737620951 I}, y = {1.4921022421124 − 2.8974295671847 I}
(1)x = {0.41396940961811}, y = {5.7074792232379}
(2)x = {0.41396940961811}, y = {5.7074792232379}
(3)x = {0.41396940961811}, y = {5.7074792232379}
(4)x = {0.26968386392917 + 0.0627897357745 I}, y = {10.0818245704855 − 3.7745009652506 I}
(5)x = {0.12062284188681 + 0.0150156649128 I}, y = {35.8022546450461 − 6.2399959914242 I}
Infinite Roots

VII.3. Case 3.

i) We have 1 form depending on the basic categorization, therefore we have the form with **exchange of the initial values** of the variables

$$3.\ \begin{cases} x \cdot y = u \wedge x = u/y \\ e^{u} = u_2 \Rightarrow u = \log(u_2) + 2k\pi i \\ y = u_1 \end{cases}$$

That like we have the Program:

Program in mathematica.

```
h2g1 [y_, u_] := N[(5 − u/y ∗ Sin[y])/ Cos[(u/y)]];
h2g2 [y_, u_] := −N [(y* log[u/y] + u/y − 6)];
g1 [y_u_] := N [y* Cos[u/y] + u/y ∗ Sin[y] − 5];
g2 [y_u_] := N [Exp[u] + y* log[u/y] + u/y − 6];
g11 [x_, y_] := N [y* Cos[x] + x* Sin[y] − 5];
g22 [x_, y_] := N [Exp [x*y] + y* log[x] + x − 6];
c := {−0.00001}; d := 10; k2 := 1;
For [k2 = −d, k2 <= d, k2 + +,
f1[u1_] := u1;
f2[u2_]:= N [log[u2] + 2 ∗ k2 ∗ I*π];
If [kl < 0, Goto [mama1],Goto [mama2]];
Label [mama1];
yrl=N[Nest[f1[h2g1[#,#]]&,N[c],4],100];
ur2 = N[Nest[f2[h2g2[#, #]]&,  N[c], 4], 100];
Goto [mama3];
Label [mama2];
yrl=N[Nest [f1[h2g1[#, #]]&, N[c], 3], 100];
ur2=N[Nest[f2[h2g2[#,#]]&,N[c],3],100];
Label [mama3];
FQ1=N[yr1, 475];
FQ2=N[ur2, 475];
s1 = N[y1, 100]/FindRoot [{y1 ∗ Cos[u1/y1] + u1/y1 ∗ Sin[y1] − 5 == 0, Exp[u1] + y1 ∗ log[u1/y1] + u1/y1−
6 == 0}, {y1, FQ2}, {u1, FQ1}, WorkingPrecision → 550];
s2 = N[u1, 100]/FindRoot [{y1 ∗ Cos[u1/y1] + u1/y1 ∗ Sin[y1] − 5 == 0, Exp[u1] + y1 ∗ log[u1/y1] + u1/y1−
6 == 0}, {y1, FQ2}, {u1, FQ1}, WorkingPrecision→ 550];
Print[”(”,k1,”)”,”x=”,N[s2/s1,14],”y=”,N[s1,14]]]
```

Result 27. For the first 11 roots with any value of $\{x, y\}$

(k2 = −1) x = {0.40935674194543 + 0.370573762095 I}, y = {1.4921022421124 − 2.8974295671847 I}
(k2 = 0) x = {−84.857112168717 + 2.9131546815141 I}, y = {−0.053124649409480 − 0.0014691954 I}

ii). The form with Normal initial values of variables for **x1 = FQ1&y1 = FQ2&{c = −1/1000}** with same program respectively we have the values.

Result 28. For the first 11 roots with any value of $\{x, y\}$

(k2 = −5)x = {1.6395769001488 − 2.5598562818337 I}, y = {6.98855656805670 − 3.462144361750 I}
(k2 = −4)x = {0.7003269200539 − 1.678528196362 I}, y = {12.6275239179040 − 3.523039064839 I}
(k2 = −3)x = {0.56485825155228 − 1.26293218945 I}, y = {12.4440052943089 − 3.302121548200 I}
(k2 = −2)x = {−0.09371084872875 − 1.1867556701 I}, y = {9.0017227411180 + 3.021208099708 I}
(k2 = −1)x = {0.58589267395229 − 0.75987830580 I}, y = {6.0322848792995 − 1.6872931285556 I}
(k2 = 0)x = {0.41396940961811}, y = {5.7074792232379}
(k2 = 1)x = {0.58589267395229 + 0.75987830580 I}, y = {6.0322848792995 + 1.6872931285556 I}
(k2 = 2)x = {−0.09371084872875 + 1.1867556701 I}, y = {9.0017227411180 − 3.0212080997081 I}
(k2 = 3)x = {0.56485825155228 + 1.26293218945 I}, y = {12.4440052943089 + 3.302121548200 I}
(k2 = 4)x = {0.7003269200539 + 1.678528196362 I}, y = {12.6275239179040 + 3.523039064839 I}
(k2 = 5)x = {1.6395769001488 + 2.559856281833 I}, y = {6.9885565680567 + 3.4621443617500 I}
Infinite Roots

IX. 9th Category Reals Roots.

We have 1 form according to the basic categorization (normal of initial values) mentioned in category 1, so

$$\left\langle\begin{array}{l}\mathrm{Sin}(y) = u \Rightarrow x = \pi - \mathrm{ArcSin}[u] + k\pi \\ x = v\end{array}\right.$$

or

$$\left\langle\begin{array}{l}\mathrm{Sin}(y) = u \Rightarrow x = \mathrm{ArcSin}[u] + k\pi \\ x = v\end{array}\right.$$

That like we have the Program:

Program in mathematica.

```
ClearAll;Off[FindRoot::jsing,FindRoot::lstol, General::ovfl,FindRoot::sszero, FindRoot::nlnum,
General::munfl,FindRoot::cvmit, General::stop];
h2g1 [x_y_] := N [(5 − y* Cos[x]) /x];
h2g2 [x_y_] := −N [(Exp [x*y] + y* log[x] − 6)];
g1[x_, y_] := N[y ∗ Cos[x] + x ∗ Sin[y] − 5];
g2 [x_, y_] := N [Exp [x*y] + y* log[x] + x − 6];
c := {0.00001}; d := 250; ff1 := 0; gg1 = 0; ff2 := 0; gg2 := 0; 1 := 0;
For[k1 = −0, k1 <= d, k1 + +,
f2[u2_] := N[π/2 − ArcSin[u2] + 2πk1];
f1[u1_] := N[u1];
gg1=0; gg2=0;
xrl = N[Nest[f2[ h2 g2[#, #]]&, −12/2, 1]];
yr2 = N[Nest[f1[ h2 g1[#, #]]&, −12/2, 1]];
FQ2 = N[xr1, 55];
FQ1 = N[yr2, 55];
s1=Im[N[x,55]/FindRoot[{y* Cos[x] + x ∗ Sin[y] − 5 == 0, Exp [x*y] + y* log[x] + x − 6 == 0},
{{x, FQ2}, {y, FQ1}},
Method→”Newton”,WorkingPrecision→ 125]];
s2 = Im [N[y, 55]/ FindRoot[{y* Cos[x] + x* Sin[y] − 5 == 0, Exp [x*y] + y* log[x] + x − 6 == 0},
{{x, FQ2}, {y, FQ1}},
Method→”Newton”, WorkingPrecision→ 125]];
m1 = N[x, 55]/FindRoot[{y ∗ Cos[x] + x ∗ Sin[y] − 5 == 0, Exp[x ∗ y] + y ∗ log[x] + x − 6 == 0},
{{x, FQ2}, {y, FQ1}},
Method>→”Newton”, WorkingPrecision → 125];
m2 = N[y, 55]/FindRoot[{y* Cos[x] + x ∗ Sin[y] − 5 == 0, Exp[x ∗ y] + y* log[x] + x − 6 == 0},
{{x, FQ2}, {y, FQ1}},
Method→”Newton”, WorkingPrecision→ 125];
```

Result 29. For the first 43 roots with any value of $\{x, y\}$

```
{k1 = 5,  Count  = 1} x = 14.96528697622235815646885,  y = −3.31344072893624394530277З
{k1 = 12,Count  = 2} x = 40.53569994772494089733913,  y = −9.328468995667519350990807
{k1 = 17, Count  = 3} x = 55.60365870303035038559625,  y = −12.34459553520115121543840
{k1 = 22, Count  = 4} x = 73.08839078039801939639899,  y = −15.63223592472422408342727
```

```
{k1 = 23, Count = 5} x = 73.51724626561076290338374, y = −15.71074618004871106108809
{k1 = 24, Count = 6} x = 74.91828882019017902838454, y = −15.96662036862422373544639
{k1 = 29, Count = 7} x = 91.70118206037697635337983, y = −18.96658473269879467434037
{k1 = 34, Count = 8} x = 108.8986068887606993064791, y = −21.93804979028911828260146
{k1 = 41, Count = 9} x = 128.8828071405692540126940, y = −25.29023402976529942020164
{k1 = 47, Count = 10} x = 146.6743423046229840254502, y = −28.20134031296098775300986
{k1 = 54, Count = 11} x = 167.3431942213169524072119, y = −31.51205483445252439145466
{k1 = 59, Count = 12} x = 185.8449925771779942986997, y = −34.42066771143056797787227
{k1 = 60, Count = 13} x = 187.0903322800541582477697, y = −34.61476879157862392669643
{k1 = 66, Count = 15} x = 206.2959754995205588383068, y = −37.58383423810846986358992
{k1 = 73, Count = 16} x = 227.7933108400394434130767, y = −40.85766159532767237440150
{k1 = 80, Count = 17} x = 249.1456034173990650705603, y = −44.06378252704148511416346
{k1 = 87, Count = 18} x = 270.8424131637692100049426, y = −47.28031003776426610933836
{k1 = 94, Count = 19} x = 292.2721680074602742568137, y = −50.42057551620268015058638
{k1 = 101, Count = 20} x = 314.1650396980154071790345, y = −53.59468004158184734623820
{k1 = 106, Count = 21} x = 333.7896627544700603390025, y = −56.41322430409156122018659
{k1 = 107, Count = 22} x = 334.2553705203098727523047, y = −56.47982107821928897655157
{k1 = 108, Count = 23} x = 335.6986038997878859756399, y = −56.68612233717420709044517
{k1 = 115, Count = 24} x = 358.0969940916112559907898, y = −59.87225574961387091050194
{k1 = 122, Count = 25} x = 380.1688689111270114828322, y = −62.98486512406748767331279
{k1 = 128, Count = 26} x = 402.6449542384080640929484, y = −66.12892732885251935537870
{k1 = 136, Count = 27} x = 424.9009255869771360055109, y = −69.21858745864134481030441
{k1 = 143, Count = 28} x = 447.3737316516577267015435, y = −72.31610928257882029248141
{k1 = 150, Count = 29} x = 469.7328638313167590435593, y = −75.37719278936515024844171
{k1 = 150, Count = 29} x = 469.7328638313167590435593, y = −75.37719278936515024844171
{k1 = 157, Count = 30} x = 492.3289319433890974798502, y = −78.45094296194890650351207
{k1 = 171, Count = 31} x = 540.7596727164887889413631, y = −84.97724474873751868693118
{k1 = 180, Count = 32} x = 563.5148991958146626420671, y = −88.01671043295848404342263
{k1 = 188, Count = 33} x = 586.3403405267661995382663, y = −91.04948359428970153314000
{k1 = 195, Count = 34} x = 609.3128184049624916732978, y = −94.08633827709201727554777
{k1 = 204, Count = 35} x = 635.3478805304329649910991, y = −97.51023261083039648281133
{k1 = 211, Count = 36} x = 658.2606162324364174341070, y = −100.5085865292655131990953
{k1 = 218, Count = 37} x = 681.4716397685880867904373, y = −103.5323845359310001282818
{k1 = 227, Count = 38} x = 707.5917782629558109875088, y = −106.9195309179959657229875
{k1 = 234, Count = 39} x = 730.7667680547884724492686, y = −109.9114949795603878423352
{k1 = 241, Count = 40} x = 754.4480803944276729265154, y = −112.9564759332630293939131
{k1 = 242, Count = 41} x = 756.0246646007884342074232, y = −113.1587644483826794631526
{k1 = 243, Count = 42} x = 756.1689717257550149455308, y = −113.1772775694306471308542
{k1 = 250, Count = 43} x = 780.4283791392372250934770, y = −116.2832790930037947556674
```

Infinite Roots

Epilogue

As we observe the system is completely solved with 9 categories of systems using the GRIM method. Of course the **total solution set** will be **the union of the solution sets** of each category separately. For the real ones we followed a separate class which gives the best and fastest solution of the system. Such systems that are complex it is necessary to analyze them in functions using their inverse in order to give us faster and more correct results.

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